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A NOTE ON THE HYPERSPACE OF FINITE SUBSETS

Abstract. In this paper, we study the relation between a space X satisfying certain generalized metric properties and its hyperspace of finite subsets $\mathcal{F}(X)$ satisfying the same properties. We prove that if $\mathcal{F}(X)$ is a stric $\mathfrak{B}_0$-space then so is X. However, there exists a stric $\mathfrak{B}_0$-space X such that $\mathcal{F}_n(X)$ is not a stric $\mathfrak{B}_0$-space for each $n \geq 2$, hence $\mathcal{F}(X)$ is not a stric $\mathfrak{B}_0$-space. Moreover, we prove that X is a P-space (resp., sequentially separable) if and only if so is $\mathcal{F}(X)$.

Key words: symmetric product, hyperspace, *sp*-network, stric $\mathfrak{B}_o$-space, P-space, sequentially separable.

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1. Introduction and preliminaries

Recently, the generalized metric properties on hyperspaces such as $\mathcal{F}(X)$, the space of finite subsets of X, and $\mathcal{F}_n(X)$, the n-fold symmetric product of X have been studied by some authors ([4], [7], [10], [11], for example). They considered several generalized metric properties and studied the relation between a space X satisfying such property and its n-fold symmetric product or its hyperspace of finite subsets satisfying the same property.

In this paper, we study the relation between a space X satisfying certain generalized metric properties and its hyperspace of finite subsets $\mathcal{F}(X)$ satisfying the same properties. We prove that if $\mathcal{F}(X)$ is a stric $\mathfrak{B}_0$-space then so is X. However, there exists a stric $\mathfrak{B}_0$-space X such that $\mathcal{F}_n(X)$ is not a stric $\mathfrak{B}_0$-space for each $n \geq 2$, hence $\mathcal{F}(X)$ is not a stric $\mathfrak{B}_0$-space. Moreover, we prove that X is a P-space (resp., sequentially separable) if and only if so is $\mathcal{F}(X)$.

Throughout this paper, all spaces are Hausdorff, $\mathbb{N}$ denotes the set of all positive integers, the first infinite ordinal denoted by ω.

Given a space X, we define its *hyperspaces* as the following sets:

(1) $CL(X) = \{A \subset X : A \text{ is closed and nonempty}\}$;

(2) $2^X = \{A \in CL(X) : A \text{ is compact}\}$;

(3) $\mathcal{F}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}$, where $n \in \mathbb{N}$;

(4) $\mathcal{F}(X) = \{A \in 2^X : A \text{ is finite}\}$.
The set $CL(X)$ is topologized by the *Vietoris topology*, the base of which consists of all subsets of the following form:

$$\mathcal{B} = \{\langle U_1, \ldots, U_k \rangle : U_1, \ldots, U_k \text{ are open subsets of } X,\ k \in \mathbb{N}\},$$

where

$$\langle U_1, \ldots, U_k \rangle = \{A \in CL(X) : A \subset \bigcup_{i \leq k} U_i,\ A \cap U_i \neq \emptyset \text{ for each } i \leq k\}.$$

Note that, by definition, 2^X, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ as the subspaces of $CL(X)$, every element of the sets 2^X, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ is nonempty. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,
(1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;
(2) 2^X is called the *hyperspace of nonempty compact subsets of* X;
(3) $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X;
(4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.
It is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

Remark 1 ([10])**.** Let X be a space and let $n \in \mathbb{N}$.
(1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.
(2) $f_n : X^n \twoheadrightarrow \mathcal{F}_n(X)$ given by $f_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\}$ is a closed finite-to-one mapping.
(3) $f_1 : X \twoheadrightarrow \mathcal{F}_1(X)$ is a homeomorphism.
(4) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$.

Notation 1 ([7])**.** *If $U_1, \ldots, U_s$ are open subsets of a space X then $\langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1, \ldots, U_s \rangle$ of the Vietoris Topology, with $\mathcal{F}(X)$.*

Notation 2. *Let X be a space. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)}$, then for each $j \leq r$, we let $U_{x_j} = \bigcap\{U \in \{U_1, \ldots, U_s\} : x_j \in U\}$. Observe that $\langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)} \subset \langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)}$.*

Definition 1 ([5])**.** *Let $\mathcal{P}$ be a family of subsets of a space X.*
(1) *The family $\mathcal{P}$ is a* network *for X, if for any neighborhood U of a point $x \in X$, there exists a set $P \in \mathcal{P}$ such that $x \in P \subset U$.*
(2) *The family $\mathcal{P}$ is an sp-*network *for X, if for each $x \in U \cap \overline{A}$ with U open and A subset in X, there is a set $P \in \mathcal{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$.*

Remark 2 ([5])**.** Base $\Longrightarrow$ *sp*-network $\Longrightarrow$ network.

Definition 2. *Let X be a space.*

(1) *The space X is said to be a* stric $\mathfrak{B}_0$-space *[5], if X is regular and has a countable sp-network.*

(2) *The space X is called a P*-space *[1], if every G_δ-set in X is open.*

(3) *The space X is said to be* sequentially separable *[2], if it has a countable sequentially dense subset. A set D is* sequentially dense *in a space X if each point $x \in X$ is the limit of some sequence of points of D.*

Definition 3. *Let X be a space.*

(1) *The space X is a* strongly Fréchet-Urysohn space *[9], if for every decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, there exist points $x_n \in A_n$ $(n \in \mathbb{N})$ such that $\{x_n : n \in \mathbb{N}\}$ converges to the point x.*

(2) *The space X is a* Fréchet-Urysohn space *[3], if for any $A \subset X$ and any $x \in \overline{A}$, there exist points $x_n \in A$ such that $\{x_n : n \in \mathbb{N}\}$ converges to x.*

(3) *The space X is called a* quasi-k-space *[8] (resp.,* k-space *[6],* sequential space *[8]), if a subset A of X is closed whenever $A \cap K$ is closed in K for every countably compact (resp., compact, compact metric) subset K of X.*

(4) *The space X is called a* k_ω-space *[6], if it is the union of countably many compact subsets K_n such that a subset A of X is closed whenever $A \cap K_n$ is closed in K_n for all $n \in \mathbb{N}$.*

Remark 3 ([3], [5], [8])**.** 1. Strongly Fréchet-Urysohn spaces $\Longrightarrow$ Fréchet-Urysohn spaces $\Longrightarrow$ sequential spaces $\Longrightarrow$ k-spaces $\Longrightarrow$ quasi-k-spaces.

2. k_ω-spaces $\Longrightarrow$ k-spaces $\Longrightarrow$ quasi-k-spaces.

Definition 4 ([5])**.** *A topological space X is called the* sequential fan*, which is denoted briefly as S_ω, if X is the quotient space by identifying all the limit points of ω many non-trivial convergent sequences.*

2. Main results

Lemma 1. *Every subspace of a stric $\mathfrak{B}_0$-space is a stric $\mathfrak{B}_0$-space.*

Proof. Let Y be a subspace of a stric $\mathfrak{B}_0$-space X. Since X is a stric $\mathfrak{B}_0$-space, X is regular and has a countable *sp*-network $\mathcal{P}$. Observe that Y is regular. If we put

$$\mathcal{G} = \{P \cap Y : P \in \mathcal{P}\},$$

then it is easy to check that $\mathcal{G}$ is a countable *sp*-network for Y. Therefore, Y is a stric $\mathfrak{B}_0$-space. ■

By Remark 1 and Lemma 1, we obtained the following theorem.

Theorem 1. *Let X be a space. If $\mathcal{F}(X)$ is a stric $\mathfrak{B}_0$-space then so is X.*

Lemma 2. *For each $n \geq 2$, $(S_\omega)^n$ is sequential but $(S_\omega)^2$ is not Fréchet -Urysohn.*

Proof. It follows from [5, Example 4.3] that S_ω is a regular Fréchet-Urysohn and k_ω-space. Therefore, $(S_\omega)^n$ is sequential for each $n \geq 2$ by Remark 3, [6, 7.5] and [8, Theorem 2.2]. Furthermore, we have $(S_\omega)^2$ is not Fréchet-Urysohn. Otherwise, since the Fréchet-Urysohn property is hereditary, $S_\omega \times (\{x_1(m) : m \in \mathbb{N}\} \cup \{x\})$ is Fréchet-Urysohn, where the sequence $\{x_1(m) : m \in \mathbb{N}\}$ converges to x in S_ω such that the set $\{x_1(m) : m \in \mathbb{N}\} \cup \{x\}$ is not discrete. By [9, Theorem 12], S_ω is strongly Fréchet-Urysohn. This is a contradiction. ■

Lemma 3 (Theorem 4.2 [5])**.** *The following conditions are equivalent for a topological space X.*

(1) *X is a k-space with a point-countable sp-network.*
(2) *X is a Fréchet-Urysohn space with a point-countable cs^*-network.*

Example 1. There exists a stric $\mathfrak{B}_0$-space X such that $\mathcal{F}_n(X)$ is not a stric $\mathfrak{B}_0$-space for each $n \geq 2$, hence $\mathcal{F}(X)$ is not a stric $\mathfrak{B}_0$-space.

Proof. For each $n \geq 2$, $(S_\omega)^n$ is sequential but $(S_\omega)^2$ is not Fréchet-Urysohn by Lemma 2. It follows from Remark 1(2) and [10, Remark 4.2, Lemma 4.4] that $\mathcal{F}_n(S_\omega)$ is sequential for each $n \geq 2$ but $\mathcal{F}_2(S_\omega)$ is not Fréchet-Urysohn. On the other hand, by Remark 1(4), $\mathcal{F}_2(S_\omega)$ is closed in $\mathcal{F}_n(S_\omega)$ for each $n > 2$. Therefore, $\mathcal{F}_n(S_\omega)$ is not Fréchet-Urysohn for each $n \geq 2$. Furthermore, it follows from [5, Example 4.3] that the sequential fan S_ω is a regular Fréchet-Urysohn space with a countable *sp*-network. This implies that it is a stric $\mathfrak{B}_0$-space. However, $\mathcal{F}_n(S_\omega)$ does not have a point-countable *sp*-network for each $n \geq 2$. Otherwise, there exists $n \geq 2$ such that $\mathcal{F}_n(S_\omega)$ has a point-countable *sp*-network. Since $\mathcal{F}_n(S_\omega)$ is sequential, $\mathcal{F}_n(S_\omega)$ is a k-space by Remark 3. It follows from Lemma 3 that $\mathcal{F}_n(S_\omega)$ is a Fréchet-Urysohn space, which is a contradiction. Therefore, $\mathcal{F}_n(S_\omega)$ does not have a point-countable *sp*-network for each $n \geq 2$. This proves that $\mathcal{F}_n(S_\omega)$ is not a stric $\mathfrak{B}_0$-space for each $n \geq 2$. By Remark 1(1) and Lemma 1, $\mathcal{F}(S_\omega)$ is not a stric $\mathfrak{B}_0$-space. ■

Lemma 4. *Let X be a space. If $\mathcal{U}$ is an open subset of $\mathcal{F}(X)$, then $\bigcup \mathcal{U}$ is an open subset of X.*

Proof. Let $\mathcal{U}$ be an open subset of $\mathcal{F}(X)$ and $x \in \bigcup\mathcal{U}$. Then, $x \in \{x, x_1, \ldots, x_r\}$ with $\{x, x_1, \ldots, x_r\} \in \mathcal{U}$. It follows from Notation 2 that we can find open subsets $U_x, U_{x_1}, \ldots, U_{x_r}$ of X such that $x \in U_x$, $x_j \in U_{x_j}$ for each $j \leq r$, and

$$\{x, x_1, \ldots, x_r\} \in \langle U_x, U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

On the other hand, if $z \in U_x$ then $\{z, x_1, \ldots, x_r\} \in \langle U_x, U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}$. Hence, $z \in \bigcup\mathcal{U}$. Thus, $U_x \subset \bigcup\mathcal{U}$. Therefore, $\bigcup\mathcal{U}$ is an open subset of X. ∎

Theorem 2. *Let X be a space. Then, X is a P-space if and only if so is $\mathcal{F}(X)$.*

Proof. *Necessity.* Let X be a P-space and $\mathcal{U}$ be a G_δ-set in $\mathcal{F}(X)$. Then, there exists a sequence $\{\mathcal{U}_m : m \in \mathbb{N}\}$ consisting of open subsets of $\mathcal{F}(X)$ such that $\mathcal{U} = \bigcap\limits_{m\in\mathbb{N}} \mathcal{U}_m$. We prove that $\mathcal{U}$ is open in $\mathcal{F}(X)$.

In fact, let $\{x_1, \ldots, x_r\} \in \mathcal{U}$. Then, $\{x_1, \ldots, x_r\} \in \mathcal{U}_m$ for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, since $\mathcal{U}_m$ is open in $\mathcal{F}(X)$, by Notation 2, there exist open subsets $U_{x_1}^{(m)}, \ldots, U_{x_r}^{(m)}$ of X such that $x_j \in U_{x_j}^{(m)}$ for each $j \leq r$, and

$$\{x_1, \ldots, x_r\} \in \langle U_{x_1}^{(m)}, \ldots, U_{x_r}^{(m)}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}_m.$$

Moreover, since $x_j \in \bigcap\limits_{m\in\mathbb{N}} U_{x_j}^{(m)}$ for each $j \leq r$, we have

$$\begin{aligned}\{x_1, \ldots, x_r\} &\in \Big\langle \bigcap_{m\in\mathbb{N}} U_{x_1}^{(m)}, \ldots, \bigcap_{m\in\mathbb{N}} U_{x_r}^{(m)}\Big\rangle_{\mathcal{F}(X)} \\ &\subset \bigcap_{m\in\mathbb{N}} \Big\langle U_{x_1}^{(m)}, \ldots, U_{x_r}^{(m)}\Big\rangle_{\mathcal{F}(X)} \subset \bigcap_{m\in\mathbb{N}} \mathcal{U}_m = \mathcal{U}.\end{aligned}$$

Since X is a P-space, $\bigcap\limits_{m\in\mathbb{N}} U_{x_j}^{(m)}$ is open in X for each $j \leq r$. It shows that the set $\langle \bigcap\limits_{m\in\mathbb{N}} U_{x_1}^{(m)}, \ldots, \bigcap\limits_{m\in\mathbb{N}} U_{x_r}^{(m)}\rangle_{\mathcal{F}(X)}$ is open in $\mathcal{F}(X)$. Hence, $\mathcal{U}$ is open in $\mathcal{F}(X)$.

Sufficiency. By definition of P-spaces, it is easy to check that P-spaces are hereditary. Hence, by Remark 1, if $\mathcal{F}(X)$ is a P-space, then X is a P-space. ∎

Theorem 3. *Let X be a space. Then, X is sequentially separable if and only if so is $\mathcal{F}(X)$.*

Proof. *Necessity.* Assume that X is sequentially separable and D is a countable sequentially dense subset of X. We put

$$\mathcal{D} = \{\{d_1, \dots, d_t\} \in \mathcal{F}(X) : d_1, \dots, d_t \in D, t \in \mathbb{N}\}.$$

Then, it is clear that $\mathcal{D}$ is a countable subset of $\mathcal{F}(X)$. Moreover, $\mathcal{D}$ is sequentially dense in $\mathcal{F}(X)$.

In fact, let $\{x_1, \dots, x_r\} \in \mathcal{F}(X)$. Then, for each $j \le r$, there exists a sequence $\{x_j^{(n)} : n \in \mathbb{N}\}$ of points of D such that $\{x_j^{(n)} : n \in \mathbb{N}\}$ converges to x_j in X. For each $n \in \mathbb{N}$, we put

$$F_n = \{x_1^{(n)}, \dots, x_r^{(n)}\}.$$

Let $\mathcal{U}$ be an open neighborhood of $\{x_1, \dots, x_r\}$ in $\mathcal{F}(X)$. By Notation 2, there exist open subsets $U_{x_1}, \dots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \le r$, and

$$\{x_1, \dots, x_r\} \in \langle U_{x_1}, \dots, U_{x_r} \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Thus, for each $j \le r$, there exists $m_j \in \mathbb{N}$ such that $x_j^{(n)} \in U_{x_j}$ for every $n \ge m_j$. If we put $m = \max\{m_j : j \le r\}$ then $F_n \in \mathcal{U}$ for every $n \ge m$. This proves that the sequence $\{F_n : n \in \mathbb{N}\}$ of points of $\mathcal{D}$ converges to $\{x_1, \dots, x_r\}$ in $\mathcal{F}(X)$.

Sufficiency. Suppose that $\mathcal{F}(X)$ is sequentially separable and $\mathcal{D}$ is a countable sequentially dense subset of $\mathcal{F}(X)$. If we put $D = \bigcup \mathcal{D}$ then D is a countable subset of X. Furthermore, D is sequentially dense in X.

In fact, let $x \in X$. Then, $\{x\} \in \mathcal{F}(X)$. Since $\mathcal{D}$ is sequentially dense in $\mathcal{F}(X)$, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ of points of $\mathcal{D}$ such that $\{F_n : n \in \mathbb{N}\}$ converges to $\{x\}$ in $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, take $x_n \in F_n$ then $x_n \in D$ and it is obvious that the sequence $\{x_n : n \in \mathbb{N}\}$ converges to x in X. ■

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