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RESULTS CONCERNING THE ANALYSIS OF GENERALIZED MITTAG-LEFFLER FUNCTION ASSOCIATED WITH EULER TYPE INTEGRALS

Abstract. In this paper, we obtain some results on certain Euler type integrals involving generalized Mittag-Leffler function defined by Salim and Faraj [20]. Further, we deduce some special cases involving Mittag-Leffler function, Wiman function, Prabhakar function, exponential, binomial and confluent hypergeometric functions. Moreover, we obtain a relation between Laguerre polynomials and Whittakar function.

Key words: generalized Mittag-Leffler function, generalized Beta function, Euler type integrals.

AMS Mathematics Subject Classification: 33C45, 47G20, 26A33.

1. Introduction and preliminaries

New extensions of some of the well-known special functions (e.g. gamma function, beta function and Gauss hypergeometric function, etc.) have been widely studied in recent past years. By interesting a regularization factor e^{-bt-1}, Chaudhary and Zubair [6] have introduced the following gamma function:

$$\Gamma_r(u) = \int_0^\infty v^{u-1}\ exp\left(-v - \frac{r}{v}\right) dv. \tag{1}$$

Chaudhary et al. [7] considered the extension of Euler's beta function as follows:

$$\Gamma_r(u, w) = \int_0^\infty v^{u-1}(1-v)^{w-1}\ exp\left(\frac{-r}{v(1-v)}\right)dv. \tag{2}$$

They also extended the Gauss's hypergeometric function [8] as follows:

$$F_r(\xi, \zeta; \lambda; u) = \sum_{n=0}^{\infty} (\xi)_n \frac{B_r(\zeta + n, \lambda - \zeta)\ u^n}{B(\zeta, \lambda - \zeta)\ n!} \tag{3}$$

$$(r \geq 0; \quad |\, u \,| < 1, \; \Re(\lambda) > \Re(\zeta) > 0),$$

where $(\xi)_n$ denotes the usual Pochhammer symbol (see [18]).

Özergin et al. [15] studied some fundamental properties of the generalized beta type function $B_r^{(\xi,\zeta,m)}(u,w)$ (see [18]):

$$B_r^{(\xi,\zeta,m)}(u,w) = \int_0^1 v^{u-1}(1-v)^{w-1}{}_1F_1\left[\xi;\zeta;\frac{-r}{v^m(1-v)^m}\right]dv \tag{4}$$

$$(\Re(\xi) > 0, \, \Re(\zeta) > 0, \, \Re(r) > 0, \, \Re(u) > 0, \, \Re(w) > 0.)$$

For $\xi = \zeta$ and $m = 1$, equation (4) reduces to equation (2).

The Mittag-Leffler function [14] was introduced in connection with its method of summations of some divergent-series. It naturally occurs as the solution of fractional ordered integral or differential equations. Many researchers (see [1]-[5], [9], [10], [11], [15], [21]) studied and defined the generalization of Mittag-Leffler function as:

Gosta Mittag-Leffler [14] introduced the function in 1903 as follows:

$$E_\xi(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(\xi n + 1)}, \qquad (\xi \in \mathbb{C}; \Re(\xi) > 0.) \tag{5}$$

A generalization of $E_\xi(u)$ was given and studied by Wiman [22] as follows:

$$E_{\xi,\zeta}(u) = \sum_{n=0}^{\infty} \frac{u^n}{\Gamma(\xi n + \zeta)}, \qquad (\Re(\xi) > 0, \Re(\zeta) > 0; \; \xi, \zeta \in \mathbb{C}.) \tag{6}$$

By means of the series representation a further generalization of $E_{\xi,\zeta}(u)$ is introduced by Prabhakar [16] as follows:

$$E_{\xi,\zeta}^{\lambda}(u) = \sum_{n=0}^{\infty} \frac{(\lambda)_n u^n}{\Gamma(\xi n + \zeta)} \quad (\Re(\lambda) > 0, \Re(\xi) > 0, \Re(\zeta) > 0; \; \lambda, \xi, \zeta \in \mathbb{C}.), \tag{7}$$

Further generalization of Mittag-Leffler function $E_{\xi,\zeta}^{\lambda,\delta}(u)$ was defined by Salim [19] as follows:

$$E_{\xi,\zeta}^{\lambda,\delta}(u) = \sum_{n=0}^{\infty} \frac{(\lambda)_n \; u^n}{\Gamma(\xi n + \zeta) \; (\delta)_n}, \tag{8}$$

where $(\xi, \zeta, \lambda, \delta \in \mathbb{C}; \; \Re(\xi) > 0, \; \Re(\zeta) > 0, \; \Re(\delta) > 0, \; \Re(\lambda) > 0)$.

A more generalized form was introduced by Salim and Faraj [20] as follows:

$$E_{\xi,\zeta,p}^{\lambda,\delta,q}(u) = \sum_{n=0}^{\infty} \frac{(\lambda)_{qn} \; u^n}{\Gamma(\xi n + \zeta)(\delta)_{pn}}, \tag{9}$$

where $\xi, \zeta, \lambda, \delta \in \mathbb{C}$; $\min\{\Re(\xi), \Re(\zeta), \Re(\lambda), \Re(\delta)\} > 0$; $p, q > 0$, $q \leq \Re(\xi + p)$ and $(\lambda)_{qn} = \frac{\Gamma(\lambda + qn)}{\Gamma(\lambda)}$ is the generalized Pochhammer symbol.

2. Main results

Theorem 1. *If* $\mu, \nu, \gamma, \delta, u, w \in \mathbb{C}$; $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\delta) > 0, \Re(u) > 0$, $\Re(w) > 0$, $\Re(p) > 0$, $\Re(q) > 0, \Re(t) > 0$; $p, q > 0$, $q \leq \Re(\mu + p)$, *then the underlying result holds true:*

$$\int_0^1 v^{u-1}(1-v)^{w-1}{}_1F_1\left[\xi;\zeta;\frac{-v}{v^m(1-v)^m}\right]E^{\lambda,\delta,q}_{\mu,\nu,p}(sv^{\mu})dv = \sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}B_t^{(\xi,\zeta,m)}(u+\mu n, w). \tag{10}$$

Proof. In order to obtain our main result, we indicate the L.H.S of (10) be denoted by I_1 and using (9), we get

$$I_1 = \int_0^1 v^{u-1}(1-v)^{w-1}{}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right]\sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n\ v^{\mu n}}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}\ dv,$$

which, on using generalized beta function in (4), yields

$$\begin{aligned} I_1 &= \sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}\int_0^1 v^{\mu n+u-1}(1-v)^{w-1}{}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right]\ dv \\ &= \sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}B_t^{(\xi,\zeta,m)}(u+\mu n, w) \end{aligned}$$

■

Corollary 1. *On setting* $\xi = \zeta$, (10) *reduces as:*

$$\int_0^1 v^{u-1}(1-v)^{w-1}\exp\left[\frac{-t}{v^m(1-v)^m}\right]\ E^{\lambda,\delta,q}_{\mu,\nu,p}(sv^{\mu})\ dv = \sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}B_t^{(m)}(u+\mu n, w). \tag{11}$$

Corollary 2. *On setting* $t = 0$, $\delta = 1$, $p = 1$, (11) *reduces as:*

$$\frac{1}{\Gamma(\xi)}\int_0^1 v^{u-1}(1-v)^{w-1}\ E^{\lambda,q}_{\mu,\nu}\ (sv^{\mu})\ dv = \sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (1)_n}B_t^{(m)}(u+\mu n, w) = E^{\lambda,q}_{\mu,\nu+w}(s). \tag{12}$$

Theorem 2. *If $\mu,\nu,\lambda,\delta,\eta,u,w\in\mathbb{C}$; $\Re(\mu)>0$, $\Re(\nu)>0$, $\Re(\lambda)>0$, $\Re(\delta)>0$, $\Re(\eta)>0$, $\Re(u)>0$, $\Re(w)>0$, $\Re(t)>0$; $|\,arg\ \frac{d+bc}{d+ac}\,|<\ \pi$; a, b, c, d are constants and $p,q>0$ then the following result holds true:*

$$(13)\quad \int_a^b (v-a)^{u-1}(b-v)^{w-1}(cv+d)^{\eta} \times {}_1F_1\left[\xi;\zeta;\frac{-t}{(v-a)^m(b-v)^m}\right]E_{\mu,\nu,p}^{\lambda,\delta,q}(s(b-v)^h)\ dv$$
$$=(ac+d)^{\eta}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\xi)_l\ (-t)^l(\lambda)_{qn}\ s^n}{(\zeta)_l\ l!\ \Gamma(\mu n+\nu)(\delta)_{pn}}B(u-lm,w+hn-lm)$$
$$\times(b-a)^{u+w+hn-2lm-1}\ {}_2F_1\left[\begin{matrix}u-lm,\ -\eta; \\ u+w+hn-2lm;\end{matrix}\ \frac{-(b-a)c}{ac+d}\right].$$

Proof. Let L.H.S of (13) be denoted by I_2. We expand the function ${}_1F_1$ into series form and using (9), we have

$$I_2=\int_a^b (v-a)^{u-1}(b-v)^{w-1}(cv+d)^{\eta}\sum_{l=0}^{\infty}\frac{(\xi)_l\ (-t)^l}{(\zeta)_l\ l!\ (v-a)^{lm}\ (b-v)^{lm}} \times\sum_{n=0}^{\infty}\frac{(\lambda)_{qn}\ s^n\ (b-v)^{hn}}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}\ dv$$
$$=\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\xi)_l\ (-t)^l\ (\lambda)_{qn}\ s^n}{(\zeta)_l\ l!\ \Gamma(\mu n+\nu)\ (\delta)_{pn}} \times\int_a^b (v-a)^{u-lm-1}(b-v)^{w+hn-lm-1}\ (cv+d)^{\eta}\ dv.$$

By using a well known formula (see [12], P. 1996 (2.7), see also [17], p-263), we get

$$\int_a^b (v-a)^{\xi-1}(b-v)^{\zeta-1}(cv+d)^{\eta}\ dv =B(\xi,\zeta)(b-a)^{\xi+\zeta-1}(ac+d)^{\eta}\ {}_2F_1\left[\xi,-\lambda;\ \xi+\zeta;\frac{-(b-a)c}{(ac+d)}\right].$$

$$(\Re(\xi)>0,\ \Re(\zeta)>0;\ \left|arg\ \frac{d+bc}{d+ac}\right|<\pi).$$

We get the desired result (13). ■

Corollary 3. *For $\xi = \zeta$, the result* (13) *reduces to*

$$(14)\quad \int_a^b (v-a)^{u-1}(b-v)^{w-1}(cv+d)^{\eta} \times \exp\left[\frac{-t}{(v-a)^m(b-v)^m}\right] E_{\mu,\nu,p}^{\lambda,\delta,q}(s(b-v)^h)\ dv$$
$$= (ac+d)^{\eta} \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-t)^l(\lambda)_{qn}\ s^n}{l!\ \Gamma(\mu n+\nu)(\delta)_{pn}}\ B(u-lm, w+hn-lm) \times (b-a)^{u+w+hn-2lm-1}\ {}_2F_1\left[\begin{matrix} u-lm,\ -\eta; \\ u+w+hn-2lm; \end{matrix}\ \frac{-(b-a)c}{ac+d}\right].$$

Corollary 4. *For $\xi = \zeta$ and $t = 0$, the result* (13) *reduces to*

$$(15)\quad \int_a^b (v-a)^{u-1}(b-v)^{w-1}(cv+d)^{\eta} E_{\mu,\nu,p}^{\lambda,\delta,q}(s(b-v)^h)\ dv$$
$$= (ac+d)^{\eta} \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)(\delta)_{pn}}\ B(u,\ w+hn) \times (b-a)^{u+w+hn-1}\ {}_2F_1\left[\begin{matrix} u,\ -\eta; \\ u+w+hn; \end{matrix}\ \frac{-(b-a)c}{ac+d}\right].$$

Corollary 5. *For $a = 0$ and $b = 1$, the result* (13) *reduces to*

$$(16)\quad \int_0^1 (v)^{u-1}(1-v)^{w-1}(cv+d)^{\eta} \times {}_1F_1\left[\xi;\zeta;\frac{-t}{(v)^m(1-v)^m}\right]\ E_{\mu,\nu,p}^{\lambda,\delta,q}(s(1-v)^h)\ dv$$
$$= (ac+d)^{\eta} \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\xi)_l\ (-t)^l(\lambda)_{qn}\ s^n}{(\zeta)_l\ l!\ \Gamma(\mu n+\nu)(\delta)_{pn}} B(u-lm, w+hn-lm) \times {}_2F_1\left[\begin{matrix} u-lm,\ -\eta; \\ u+w+hn-2lm; \end{matrix}\ \frac{-c}{d}\right].$$

Theorem 3. *If $\mu,\nu,\lambda,\delta,\alpha,\gamma,u,w \in \mathbb{C}$; $\Re(\mu) > 0$, $\Re(\nu) > 0$, $\Re(\lambda) > 0$, $\Re(\delta) > 0$, $\Re(\alpha) > 0$, $\Re(\gamma) > 0$, $\Re(u) > 0$, $\Re(w) > 0$, $\Re(t) > 0$, $p,q > 0$, then the following result holds true:*

$$(17)\quad \int_0^1 (v)^{u-1}(1-v)^{w-u-1}(1-zv^{\alpha}(1-v)^{\gamma})^{-a} \times {}_1F_1\left[\xi;\zeta;\frac{-t}{(v)^m(1-v)^m}\right] E_{\mu,\nu,p}^{\lambda,\delta,q}(sv^{\mu})\ dv$$
$$= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_l\ z^l(\lambda)_{qn}\ s^n}{l!\ \Gamma(\mu n+\nu)\ (\delta)_{pn}}\ B_t^{(\xi,\ \zeta,\ m)}(u+\mu n+\alpha l,\ w+\gamma l-u).$$

Proof. For suitability, let L.H.S of (17) be denoted by I_3. Applying (9) and expressing $(1 - zv^\alpha(1 - v)^\gamma)^{-a}$ into binomial series (see [18]), we get

$$\begin{aligned} I_3 &= \int_0^1 (v)^{u-1}(1-v)^{w-u-1}(1 - zv^\alpha(1-)v^\gamma)^{-a}{}_1F_1\left[\xi;\zeta;\frac{-t}{(v)^m(1-v)^m}\right] \\ &\qquad \times \sum_{l=0}^{\infty} \frac{(a)_l\ z^l\ v^{\alpha l}\ (1-v)^{\gamma l}}{l!} \sum_{n=0}^{\infty} \frac{(\lambda)_{qn}\ s^n\ v^{\mu n}}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}\ dv \\ &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_l\ z^l(\lambda)_{qn}\ s^n}{l!\ \Gamma(\mu n+\nu)\ (\delta)_{pn}} \int_0^1 (v)^{u+\alpha l+\mu n-1}(1-v)^{w+\gamma l-u-1} \\ &\qquad \times {}_1F_1\left[\xi;\zeta;\frac{-t}{(v)^m(1-v)^m}\right] dv \\ &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_l\ z^l(\lambda)_{qn}\ s^n}{l!\ \Gamma(\mu n+\nu)\ (\delta)_{pn}}\ B_t^{(\xi,\ \zeta,\ m)}(u+\mu n+\alpha l,\ w+\gamma l-u) \end{aligned}$$

which, upon using (4), yields our desired result (17). ■

Corollary 6. *For $\xi = \zeta$ in* (17)*, and then using the integral formula* [*see [12], Equation (3.5)*] *we obtain the underlying result:*

$$\begin{aligned} \int_0^1 (v)^{u-1}(1-v)^{w-u-1}(1 - xv^\alpha(1-v)^\gamma)^{-a}\ \exp\left[\frac{-t}{(v)^m(1-v)^m}\right] dv \\ = \sum_{r=0}^{\infty} \frac{(a)_r\ x^r}{r!}\ B_t^{(m)}(u+\alpha r,\ w+\gamma r-u) \end{aligned}$$

thus, (17) *reduces to*

$$\begin{aligned} (18)\qquad & \int_0^1 (v)^{u-1}(1-v)^{w-u-1}(1 - xv^\alpha(1-v)^\gamma)^{-a} \\ &\qquad \times \exp\left[\frac{-t}{(v)^m(1-v)^m}\right] E_{\mu,\nu,p}^{\lambda,\delta,q}(sv^\mu)\ dv \\ &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_l\ x^l(\lambda)_{qn}\ s^n}{l!\ \Gamma(\mu n+\nu)\ (\delta)_{pn}}\ B_t^{(m)}(u+\mu n+\alpha l,\ w+\gamma l-u). \end{aligned}$$

Corollary 7. *For $\xi = \zeta$ and $a = 0$,* (17) *reduces as follows:*

$$\begin{aligned} (19)\qquad & \int_0^1 (v)^{u-1}(1-v)^{w-u-1} \exp\left[\frac{-t}{(v)^m(1-v)^m}\right] E_{\mu,\nu,p}^{\lambda,\delta,q}(sv^\mu)\ dv \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_{pn}}\ B_t^{(m)}(u+\mu n,\ w-u), \end{aligned}$$

where $B_t^{(m)}$ is the extended beta function given by (2).

Relationship with Laguerre Polynomial and Whittaker function:

Taking $\xi = \zeta$, $t = \beta$, $a = 0$ and considering the following equality:

$$e^{\frac{-\beta}{v(1-v)}} = e^{\frac{-\beta}{1-v}} \; e^{\frac{-\beta}{v}}$$

and then using the definition (9) and the following generating function of the Laguerre polynomial (see [18])

$$e^{\frac{-\beta}{v(1-v)}} = e^{-\beta} e^{\frac{-\beta}{v}} (1-v) \sum_{r=0}^{\infty} L_r(\beta) v^r,$$

in (17), we get

$$I_3 = \int_0^1 (v)^{u-1} (1-v)^{w-u-1} \; e^{-\beta} \; e^{\frac{-\beta}{v(1-v)}} \times \sum_{r=0}^{\infty} L_r(\beta) \; v^r \sum_{n=0}^{\infty} \frac{(\lambda)_{qn} \; s^n \; v^{\mu n}}{\Gamma(\mu n + \nu) \; (\delta)_{pn}} \, dv \tag{20}$$

$$= e^{-\beta} \sum_{r,n=0}^{\infty} \frac{L_r(\beta) \; (\lambda)_{qn} \; s^n}{\Gamma(\mu n + \nu) \; (\delta)_{pn}} \int_0^1 (v)^{u+\mu n + r - 1} (1-v)^{w-u-1} \; e^{\frac{-\beta}{v}} \, dv \tag{21}$$

finally, using the following integral representation [see [13], P. 25(2.2)]

$$\int_0^1 (v)^{u-1} (1-v)^{w-1} \; e^{\frac{-\beta}{v}} \, dv = \Gamma(w) \; \beta^{\frac{u-1}{2}} e^{\frac{-\beta}{2}} \; W_{\frac{(1-u-2w)}{2}, \; \frac{u}{2}} (\beta),$$

$\Re(w) > 0$, $\Re(\beta) > 0$ in equation (21), we get

$$\begin{aligned} I_3 \; &= \; e^{-\beta} \sum_{r,n=0}^{\infty} \frac{L_r(\beta) \; (\lambda)_{qn} \; s^n}{\Gamma(\mu n + \nu) \; (\delta)_{pn}} \Gamma(w-u) \\ &\qquad \times \beta^{\frac{u+\mu n + r - 1}{2}} e^{\frac{-\beta}{2}} \; W_{\frac{(1+u-\mu n - r - 2w)}{2}, \; \frac{u+\mu n + r}{2}} (\beta) \\ &= \; e^{\frac{-3\beta}{2}} \sum_{r,n=0}^{\infty} \frac{L_r(\beta) \; (\lambda)_{qn} \; s^n}{\Gamma(\mu n + \nu) \; (\delta)_{pn}} \Gamma(w-u) \\ &\qquad \times \beta^{\frac{u+\mu n + r - 1}{2}} \; W_{\frac{(1+u-\mu n - r - 2w)}{2}, \; \frac{u+\mu n + r}{2}} (\beta), \end{aligned} \tag{22}$$

where $L_r(\beta)$ is Laguerre polynomial and $W_{\mu,\nu}(\beta)$ is Whittaker function.

3. Special cases

(i) On replacing $p = q = 1$ in (10), we have

$$\int_0^1 v^{u-1}(1-v)^{w-1} {}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right] E_{\mu,\nu}^{\lambda,\delta}(sv^{\mu})\, dv = \sum_{n=0}^{\infty} \frac{(\lambda)_n\ s^n}{\Gamma(\mu n+\nu)\ (\delta)_n} B_t^{(\xi,\zeta,m)}(u+\mu n, w), \tag{23}$$

where $E_{\mu,\nu}^{\lambda,\delta}(s)$ is a Mittag-Leffler function defined in (see [17]).

(ii) On replacing $p = \delta = 1$ in (10), we have

$$\int_0^1 v^{u-1}(1-v)^{w-1} {}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right] E_{\mu,\nu}^{\lambda,q}(sv^{\mu})\, dv = \sum_{n=0}^{\infty} \frac{(\lambda)_{qn}\ s^n}{\Gamma(\mu n+\nu)\ n!} B_t^{(\xi,\zeta,m)}(u+\mu n, w), \tag{24}$$

where $E_{\mu,\nu}^{\lambda,\delta}(s)$ is a Mittag-Leffler function defined in (see [18]).

(iii) On replacing $p = q = \delta = 1$ in (10), we have

$$\int_0^1 v^{u-1}(1-v)^{w-1} {}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right] E_{\mu,\nu}^{\lambda}(sv^{\mu})\, dv = \sum_{n=0}^{\infty} \frac{(\lambda)_n\ s^n}{\Gamma(\mu n+\nu)\ n!} \quad B_t^{(\xi,\zeta,m)}(u+\mu n, w), \tag{25}$$

where $E_{\mu,\nu}^{\lambda}(s)$ is a Mittag-Leffler function defined in (see [16]).

(iv) On replacing $p = q = \lambda = \delta = 1$ in (10), we have

$$\int_0^1 v^{u-1}(1-v)^{w-1} {}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right] E_{\mu,\nu}(sv^{\mu})\, dv = \sum_{n=0}^{\infty} \frac{s^n}{\Gamma(\mu n+\nu)} B_t^{(\xi,\zeta,m)}(u+\mu n, w), \tag{26}$$

where $E_{\mu,\nu}(s)$ is a Wiman function defined in (see [22]).

(v) On replacing $\mu = \nu = p = q = \lambda = \delta = 1$ in (10), we have

$$\int_0^1 v^{u-1}(1-v)^{w-1} {}_1F_1\left[\xi;\zeta;\frac{-t}{v^m(1-v)^m}\right] e^{(sv)}\, dv = \sum_{n=0}^{\infty} \frac{(sv)^n}{n!} B_t^{(\xi,\zeta,m)}(u+n, w), \tag{27}$$

where $e^{(s)}$ is a exponential function.

(vi) On replacing $\nu = \delta = p = q = 1$, $\mu = 0$ in (13), we have

$$(28)\quad \int_a^b (v-a)^{u-1}(b-v)^{w-1}(cv+d)^{\eta} \times {}_1F_1\left[\xi;\zeta;\frac{-t}{(v-a)^m(b-v)^m}\right] {}_1F_0(\lambda;-;s(b-v)^h)\, dv$$
$$= (ac+d)^{\eta}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\xi)_l\ (-t)^l(\lambda)_n\ s^n}{(\zeta)_l\ l!\ n!}B(u-lm, w+hn-lm) \times (b-a)^{u+w+hn-2lm-1}\ {}_2F_1\left[\begin{matrix} u-lm,\ -\eta; \\ u+w+hn-2lm; \end{matrix}\ \frac{-(b-a)c}{ac+d}\right],$$

where ${}_1F_0$ is a Binomial function defined in (see [18]).

(vii) On replacing $\mu = \nu = \lambda = \delta = p = q = 1$, $a = t = 0$ in (17), we have

$$(29)\qquad \int_0^1 e^{sv}(v)^{u-1}(1-v)^{w-u-1}\, dv = \frac{\Gamma(u)\,\Gamma(w-u)}{\Gamma(w)}\ {}_1F_1(u;w;s),$$

where $\Re(w) > 0$, $\Re(u) > 0$ and ${}_1F_1$ is a confluent hypergeometric function defined in (see [18]).

4. Conclusion

The results obtained in this paper are general in nature and can be specialized to compute numerous known and new integral formulas with several special functions from the close relationship of generalized Mittag-Leffler functions. We also connect our main results with Laguerre polynomials and Whittakar function.

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