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Marek Miarka

AN OTHER PROOF OF THE REICH FIXED POINT THEOREM

Abstract. We give a simple and nonconstructive proof of the Reich fixed point theorem which generalizes both Banach and Kannan fixed point theorems.

Key words: fixed point, complete metric space, the Reich fixed point theorem.

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1. Introduction

In 1968, an interesting fixed point theorem was put forward by Kannan [3]. His theorem is important because on one hand, it generalizes the famous Banach's fixed point theorem, and on the other hand, as shown by Sybrahmanyam [5], it characterizes complete metric spaces. In 1971, Reich [4] proved the following generalization of Kannan's result.

Theorem 1. *Let (X, d) be a complete metric space. Let $f\colon X \longrightarrow X$ be a mapping satisfying the following condition:*

$$d(f(x), f(y)) \leq Ad(x, f(x)) + Bd(y, f(y)) + Cd(x, y) \quad (x, y \in X) \tag{1}$$

where $A, B, C \in [0, \infty[$ satisfy $A + B + C < 1$. Then f has a unique fixed point.

Notice that $A = B = 0$, $C \in]0, 1[$ gives the Banach fixed point theorem. Moreover, $A = B \in]0, \frac{1}{2}[$, $C = 0$ yields the Kannan fixed point theorem (see [3] or [2]).

In this note we give a new proof of Reich's theorem, inspired by the nonconstructive proof of the Banach fixed point theorem (see for instance [1, Thm. 2.1]).

2. Proof of Reich's theorem

Proof. Define

$$\alpha = \inf\{d(z, f(z))\colon z \in X\}\,. \tag{2}$$

We will show that $\alpha = 0$. Suppose, on the contrary, that $\alpha > 0$. Then for each $\varepsilon > 0$ there exists $z_\varepsilon \in X$ with

$$d(z_\varepsilon, f(z_\varepsilon)) < \alpha + \varepsilon\,. \tag{3}$$

In view of (1), we have

$$d(f(x), f^2(x)) \le Ad(x, f(x)) + Bd(f(x), f^2(x)) + Cd(x, f(x)) \quad (x \in X)\,. \tag{4}$$

From (2), (3) and (4) applied to $x = z_\varepsilon$ we get $(1 - B)\alpha \le (A + C)(\alpha + \varepsilon)$. Letting $\varepsilon \longrightarrow 0+$ we obtain a contradiction. Hence $\alpha = 0$.

Define

$$F_\varepsilon = \mathrm{cl}\{x \in X\colon d(x, f(x)) \le \varepsilon\}(\ne \emptyset) \quad (\varepsilon > 0)\,. \tag{5}$$

For each $\varepsilon > 0$ fix $x_{0,\varepsilon} \in F_\varepsilon$. Hence for every $\varepsilon > 0$ there exists a sequence

$$\{x_{n,\varepsilon}\}_{n=1}^{\infty} \subset \{x \in X\colon d(x, f(x)) \le \varepsilon\} \tag{6}$$

such that

$$\lim_{n\to\infty} d(x_{n,\varepsilon}, x_{0,\varepsilon}) = 0. \tag{7}$$

From (1) and the triangle inequality we obtain

$$\begin{aligned} d(x_{0,\varepsilon}, f(x_{0,\varepsilon})) &\le d(x_{0,\varepsilon}, x_{n,\varepsilon}) + d(x_{n,\varepsilon}, f(x_{0,\varepsilon})) \\ &\le d(x_{0,\varepsilon}, x_{n,\varepsilon}) + d(x_{n,\varepsilon}, f(x_{n,\varepsilon})) + d(f(x_{n,\varepsilon}), f(x_{0,\varepsilon})) \\ &\le d(x_{0,\varepsilon}, x_{n,\varepsilon}) + d(x_{n,\varepsilon}, f(x_{n,\varepsilon})) + Ad(x_{n,\varepsilon}, f(x_{n,\varepsilon})) \\ &\qquad + Bd(x_{0,\varepsilon}, f(x_{0,\varepsilon})) + Cd(x_{n,\varepsilon}, x_{0,\varepsilon}) \\ &\le (1 + C)d(x_{0,\varepsilon}, x_{n,\varepsilon}) + (1 + A)d(x_{n,\varepsilon}, f(x_{n,\varepsilon})) \\ &\qquad + Bd(x_{0,\varepsilon}, f(x_{0,\varepsilon})) \end{aligned}$$

for $n \in \mathbb{N}$ and $\varepsilon > 0$. Combining it with (6) and (7), and letting $n \to \infty$,

$$d(x_{0,\varepsilon}, f(x_{0,\varepsilon})) \le \frac{1 + A}{1 - B}\varepsilon \quad (\varepsilon > 0). \tag{8}$$

Let $u_\varepsilon, w_\varepsilon \in F_\varepsilon \quad (\varepsilon > 0)$. Combining the triangle inequality with (1) and (8) yields

$$\begin{aligned} d(u_\varepsilon, w_\varepsilon) &\leq d(u_\varepsilon, f(u_\varepsilon)) + d(f(u_\varepsilon), w_\varepsilon) \\ &\leq d(u_\varepsilon, f(u_\varepsilon)) + d(f(u_\varepsilon), f(w_\varepsilon)) + d(f(w_\varepsilon), w_\varepsilon) \\ &\leq d(u_\varepsilon, f(u_\varepsilon)) + d(w_\varepsilon, f(w_\varepsilon)) + Ad(u_\varepsilon, f(u_\varepsilon)) \\ &\qquad + Bd(w_\varepsilon, f(w_\varepsilon)) + Cd(u_\varepsilon, w_\varepsilon) \\ &\leq (1+A)d(u_\varepsilon, f(u_\varepsilon)) + (1+B)d(w_\varepsilon, f(w_\varepsilon)) + Cd(u_\varepsilon, w_\varepsilon) \\ &\leq (1+A)\frac{1+A}{1-B}\varepsilon + (1+B)\frac{1+A}{1-B}\varepsilon + Cd(u_\varepsilon, w_\varepsilon). \end{aligned}$$

This implies that

$$\lim_{\varepsilon \to 0+} \operatorname{diam} F_\varepsilon = 0. \tag{9}$$

Applying (2), (5), (9) and and the completness of (X, d) we see that $\{F_\varepsilon \colon \varepsilon > 0\}$ satisfies assumptions of the Cantor interesection Theorem.

Finally, $\bigcap \{F_\varepsilon \colon \varepsilon > 0\} = \{x_f\}$ where x_f is a unique fixed point of f. ■

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MAREK MIARKA
INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSAW, POLAND
e-mail: m.miarka@uw.edu.pl

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