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ON ENESTRÖM-KAKEYA THEOREM FOR A QUATERNIONIC POLYNOMIAL

Abstract. Let $p(q) = \sum_{l=0}^{n} q^l a_l$ be a quaternion polynomial of degree n with quaternion coefficients a and quaternion variable q, where $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ for $0 \le l \le n$. In this paper, we put some restrictions on the coefficients of $p(q)$ to obtain some new Eneström-Kakeya's Theorems for a polynomial with quaternion variable.

Key words: Eneström-Kakeya theorem, zeros of polynomials, quaternions.

AMS Mathematics Subject Classification: 30C10, 30E10, 30C15.

1. Introduction

Let $p(z) = \sum_{l=0}^{n} a_l z^l$ be a complex polynomial of degree n. One of the fundamental problem of finding out the region which contains all or a prescribed number of zeros of a polynomial was first studied by Gauss [6]. He proved:

Theorem 1. *If $p(z) = z^n + \sum_{l=1}^{n-1} a_l z^l$, where a_l are all real, then $p(z)$ has all its zeros in $|z| \le R$, where (i) $R = \max(1, 2^{\frac{1}{2}} s)$, s being the sum of positive a_l (ii) $R = \max(n 2^{\frac{1}{2}} |a_l|)^l$.*

In 1829, Cauchy [2] gave more exact bounds for the moduli of zeros of a polynomial than those obtained by Gauss [6]. He proved the following.

Theorem 2. *All the zeros of the polynomial $p(z) = \sum_{l=0}^{n} a_l z^l$ of degree n lie in the circle $|z| \le R$, where R is a root of the equation*

$$|a_0| + |a_1|z + |a_2|z^2 + ... + |a_{n-1}|z^{n-1} + |a_n|z^n = 0.$$

Several generalisations and improvements of these two results are available in the literature (see [3, 4, 9, 10, 12]). The following well known result on the location of zeros of a complex polynomial with restricted coefficients is referred to in the literature as the Eneström-Kakeya Theorem [11].

Theorem 3 (Eneström-Kakeya)**.** *Let $p(z) = \sum_{l=0}^{n} a_l z^l$ be a polynomial of degree n (where z is a complex variable) whose coefficients a_l satisfy*

$$a_n \geq a_{n-1} ... \geq a_1 \geq a_0 > 0,$$

then all the zeros of $p(z)$ lie in the closed unit disk $|z| \leq 1$.

Joyal, Labella and Rahman [9] extended Theorem 3 to a polynomial whose coefficients are monotonic but need not be non-negative. Infact, they obtained the following:

Theorem 4. *Let $p(z) = \sum_{l=0}^{n} a_l z^l$ be a complex polynomial of degree n such that*

$$a_n \geq a_{n-1} ... \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{a_n + |a_0| - a_0}{|a_n|}.$$

Let $\mathbb{P}_n := \{p : p(q) = \sum_{l=0}^{n} q^l a_l, q \in \mathbb{H}\}$ denote the class of n^{th}-degree polynomials with quaternion variable $q \in \mathbb{H}$ and a_l, $0 \leq l \leq n$ are either real or quaternion. The analytic theory of functions of a quaternion variable $\mathbb{H}$ as a number system, which extends the complex number has been developed recently for distribution of zeros of quaternion polynomials and convergent power series by several authors. The well known notation $\mathbb{H} = \{\alpha + i\beta + j\gamma + k\delta | \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ where i, j, k are the usual directional units such that $i^2 = j^2 = k^2 = ijk = -1$. It is easy to see that any two given quaternion is no commutative.

Carney et al. [1] extended the Eneström-Kakeya Theorem from complex polynomials to quaternion polynomials, and then proved the following result.

Theorem 5. *If $p(q) = \sum_{l=0}^{n} q^l a_l$ is a polynomial of degree n(where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} ... \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of p lie in $|q| \leq 1$.

They [1] also obtained the following generalization of some existing results for a quaternion polynomial with monotone increasing real parts and imaginary parts.

Corollary 1. *If $p(q) = \sum_{l=0}^{n} q^l a_l$ is a polynomial of degree n with quaternion coefficients and quaternionic variable, where $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ for $0 \leq l \leq n$, satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_0, \; \beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_0, \; \gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_0,$$

$$\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_0, \;\; 0 \leq l \leq n,$$

then all the zeros of p *lie in*

$$|q| \;\leq\; \frac{1}{|a_n|}\big[(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 - \delta_n)\big].$$

For recent results on these, interested readers can see e.g., Carney et al. [1], Gentili and Stoppato [7], Gentili and Struppa [8], Pogorui and Shapiro [13], and the references contained in them.

It is the purpose of this paper, following the style of [1] to extend Eneström-Kakeya Theorem in the complex setting to some new Eneström-Kakeya Theorems in quaternion variables.

2. Main results

Theorem 6. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternion coefficients* a *and quaternion variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that for some* $0 \leq r \leq n$*, we have*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_r \leq \alpha_{r-1} \leq \cdots \leq \alpha_0,$$

and for some $0 \leq s \leq n$

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_s \leq \beta_{s-1} \leq \cdots \leq \beta_0,$$

and for some $0 \leq u \leq n$

$$\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_u \leq \gamma_{u-1} \leq \cdots \leq \gamma_0,$$

and for some $0 \leq v \leq n$

$$\delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_v \leq \delta_{v-1} \leq \cdots \leq \delta_0.$$

Then all the zeros of $p(q)$ *lie in*

$$|q| \;\leq\; \frac{1}{|a_n|}\big\{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M\big\},$$

where

$$M = \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}| + \sum_{l=1}^{s} |\beta_l - \beta_{l-1}| + \sum_{l=1}^{u} |\gamma_l - \gamma_{l-1}| + \sum_{l=1}^{v} |\delta_l - \delta_{l-1}|.$$

Proof. Define f as in the proof of Theorem 1.6 of [1] as $f(q) = p(q) * (1-q) + q^{n+1}a_n$.

$$\begin{aligned} f(q) &= (\alpha_0 + i\beta_0 + j\gamma_0 + k\delta_0) + \sum_{l=1}^{n} q^l(\alpha_l - \alpha_{l-1}) \\ &\quad + i\sum_{l=1}^{n} q^l(\beta_l - \beta_{l-1}) + j\sum_{l=1}^{n} q^l(\gamma_l - \gamma_{l-1}) \\ &\quad + k\sum_{l=1}^{n} q^l(\delta_l - \delta_{l-1}). \end{aligned}$$

For $|q| = 1$, we have

$$\begin{aligned} |f(q)| &\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{l=1}^{n} |\alpha_l - \alpha_{l-1}| \\ &\quad + \sum_{l=1}^{n} |\beta_l - \beta_{l-1}| + \sum_{l=1}^{n} |\gamma_l - \gamma_{l-1}| + \sum_{l=1}^{n} |\delta_l - \delta_{l-1}| \\ &= (|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0|) + \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}| \\ &\quad + \sum_{l=r+1}^{n} (\alpha_l - \alpha_{l-1}) + \sum_{l=1}^{s} |\beta_l - \beta_{l-1}| \\ &\quad + \sum_{l=s+1}^{n} (\beta_l - \beta_{l-1}) + \sum_{l=1}^{u} |\gamma_l - \gamma_{l-1}| \\ &\quad + \sum_{l=u+1}^{n} (\gamma_l - \gamma_{l-1}) + \sum_{l=1}^{v} |\delta_l - \delta_{l-1}| + \sum_{l=v+1}^{n} (\delta_l - \delta_{l-1}) \\ &= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| - \alpha_r - \beta_s - \gamma_u - \delta_v + \beta_n + \gamma_n + \delta_n + M. \end{aligned}$$

For $q^n * f(1/q) = q^n * \sum_{l=0}^{n} q^{-l}a_l = \sum_{l=0}^{n} q^{n-l}a_l$, we have

$$\max_{|q|=1} |q^n * f(1/q)| = \max\nolimits_{|q|=1} |f(1/q)| = \max_{|q|=1} |f(q)|,$$

implying that $q^n * f(1/q)$ have same bound on $|q| = 1$. By Max. modulus Theorem [1, Theorem 8], we see that, for $|q| \leq 1$

$$\begin{aligned} |q^n * f(1/q)| &\leq ((|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M), \end{aligned}$$

implying that

$$\begin{aligned} |f(1/q)| &\leq \frac{1}{|q|^n}((|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M). \end{aligned}$$

By replacing q by $1/q$ in the last inequality, we get

$$\begin{aligned}|f(q)| \leq &((|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ &+ (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M)|q|^n.\end{aligned}$$

Observe that, for $|q| \geq 1$

$$\begin{aligned}|p(q) * (1-q)| &\geq |a_n||q|^{n+1} - |f(q)| \\ &\geq |a_n||q|^{n+1} - ((|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M)|q|^n \\ &= (|a_n||q| - ((|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ &\quad + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M))|q|^n.\end{aligned}$$

So if

$$\begin{aligned}|q| > &\frac{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) + (|\gamma_0| - \gamma_u + \gamma_n)}{|a_n|} \\ &+ \frac{(|\delta_0| - \delta_v + \delta_n) + M}{|a_n|}\end{aligned}$$

(in which case $|q| \geq 1$) then $|p(q) * (1-q)| > 0$ and $p(q) * (1-q) \neq 0$. Since the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of p, then for

$$\begin{aligned}|q| > &\frac{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) + (|\gamma_0| - \gamma_u + \gamma_n)}{|a_n|} \\ &+ \frac{(|\delta_0| - \delta_v + \delta_n) + M}{|a_n|}\end{aligned}$$

we have $p(q) \neq 0$. That is, all the zeros of p lie in

$$\begin{aligned}|q| \leq &\frac{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) + (|\gamma_0| - \gamma_u + \gamma_n)}{|a_n|} \\ &+ \frac{(|\delta_0| - \delta_v + \delta_n) + M}{|a_n|}\end{aligned}$$

This completes the proof of Theorem 6. ■

Remark 1. (a) Notice that if we let $\beta_l = \gamma_l = \delta_l = 0$ for $l = 0, 1, 2, \cdots, n$ and $r = 0$ in Theorem 6 then we recapture Theorem 8 of [1] as corollary, which in turn is a significant generalizations of results in [3, 4, 5, 6].

(b) Theorem 6 gives several corollaries with hypotheses involving the quaternion variable parts. For example, with $r = s = u = v = n$, we have the hypothesis that

$$\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_0, \ \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_0,$$

$$\gamma_n \leq \gamma_{n-1} \leq \cdots \leq \gamma_0, \ \delta_n \leq \delta_{n-1} \leq \cdots \leq \delta_0.$$

Consequently, we have the following corollaries.

Corollary 2. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternion coefficients* a *and quaternionic variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that we have*

$$\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_0,\ \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_0,$$

$$\gamma_n \leq \gamma_{n-1} \leq \cdots \leq \gamma_0,\ \delta_n \leq \delta_{n-1} \leq \cdots \leq \delta_0.$$

Then all the zeros of $p(q)$ *lie in*

$$\begin{aligned} |q| \ \leq \ & \frac{1}{|a_n|}\{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) \\ & + (|\gamma_0| - \gamma_u + \gamma_n) + (|\delta_0| - \delta_v + \delta_n) + M\}, \end{aligned}$$

where

$$M = \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}| + \sum_{l=1}^{s} |\beta_l - \beta_{l-1}| + \sum_{l=1}^{p} |\gamma_l - \gamma_{l-1}| + \sum_{l=1}^{q} |\delta_l - \delta_{l-1}|.$$

With $r = s = u = v = 0$, Theorem 2.1 gives:

Corollary 3. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternionic coefficients* a *and quaternionic variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that we have*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_0, \beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_0,$$

$$\gamma_n \geq \gamma_{n-1} \geq \cdots \geq \gamma_0, \delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_0.$$

Then all the zeros of $p(q)$ *lie in*

$$\begin{aligned} |q| \ \leq \ & \frac{1}{|a_n|}\big\{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) \\ & + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)\big\}. \end{aligned}$$

Assume we let $r = n$ and ($s = u = v = 0$ or $r = 0$ and $s = u = v = n$), Theorem 2.1 gives the next two results.

Corollary 4. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternion coefficients* a *and quaternion variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that we have*

$$\alpha_0 \geq \cdots \geq \alpha_{n-1} \geq \alpha_n,\ \beta_0 \leq \cdots \leq \beta_{n-1} \leq \beta_n,$$

$$\gamma_0 \leq \cdots \leq \gamma_{n-1} \leq \gamma_n,\ \delta_0 \leq \cdots \leq \delta_{n-1} \leq \delta_n.$$

Then all the zeros of $p(q)$ *lie in*

$$|q| \ \leq \ \frac{1}{|a_n|}\{(|\alpha_0| + M) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)\},$$

where

$$M = \sum_{l=1}^{n} |\alpha_l - \alpha_{l-1}|.$$

Corollary 5. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternion coefficients* a *and quaternion variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that we have*

$$\alpha_0 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n,\ \beta_0 \geq \cdots \geq \beta_{n-1} \geq \beta_n,$$

$$\gamma_0 \geq \cdots \geq \gamma_{n-1} \geq \gamma_n,\ \delta_0 \geq \cdots \geq \delta_{n-1} \geq \delta_n.$$

Then all the zeros of $p(q)$ *lie in*

$$|q| \leq \frac{1}{|a_n|}\big\{(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| + |\gamma_0| + |\delta_0| + M\big\},$$

where

$$M = \sum_{l=1}^{n}(|\beta_l - \beta_{l-1}| + |\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|).$$

Theorem 7. *Let* $p(q) = \sum_{l=0}^{n} q^l a_l$ *be a polynomial of degree* n *with quaternion coefficients* a *and quaternion variable* q*, where* $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ *for* $0 \leq l \leq n$*. Suppose that for some* $0 \leq r \leq n-1$*, we have*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_r.$$

Then all the zeros of $p(q)$ *lie in*

$$q| \leq \frac{1}{|a_n|}\big\{(|\alpha_0| - \alpha_r + \alpha_n) + |\beta_0| + |\gamma_0| + |\delta_0| + L + M\big\},$$

where

$$L = \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}| \text{ and } M = \sum_{l=1}^{n}(|\beta_l - \beta_{l-1}| + |\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|).$$

Proof. As before, let $f(q) = p(q) \star (1-q) + q^{n+1}a_n$. For $|q| = 1$, we have

$$\begin{aligned}
|f(q)| &= |a_0 + \sum_{l=1}^{n} q^l (a_l - a_{l-1})| \leq |a_0| + \sum_{l=1}^{n} |a_l - a_{l-1}| \\
&\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{l=1}^{n} |\alpha_l - \alpha_{l-1}| \\
&\quad + \sum_{l=1}^{n} (|\beta_l - \beta_{l-1}| + |\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|) \\
&= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}| \\
&\quad + \sum_{l=r+1}^{n} (\alpha_l - \alpha_{l-1}) + \sum_{l=1}^{n} (|\beta_l - \beta_{l-1}| + |\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|) \\
. &= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| - \alpha_r + \alpha_n + L + M.
\end{aligned}$$

Now we follow the style of the proof of Theorem 2.1 for $|q| \geq 1$ and obtain

$$|q| > \frac{(|\alpha_0| - \alpha_r + \alpha_n) + |\beta_0| + |\gamma_0| + |\delta_0| + L + M}{|a_n|}.$$

Finally

$$\begin{aligned}
|q| \leq & \frac{(|\alpha_0| - \alpha_r + \alpha_n) + (|\beta_0| - \beta_s + \beta_n) + (|\gamma_0| - \gamma_u + \gamma_n)}{|a_n|} \\
& + \frac{(|\delta_0| - \delta_v + \delta_n) + M}{|a_n|}
\end{aligned}$$

and the proof of Theorem 7 is complete. ■

Set $\beta_l = \gamma_l = \delta_l = 0 \ \forall l = 0, 1, 2, \cdots, n$ in Theorem 7 to have the following corollary .

Corollary 6. *Let $p(q) = \sum_{l=0}^{n} q^l a_l$ be a polynomial of degree n with quaternion coefficients a and quaternion variable q for $0 \leq l \leq n$. Suppose that for some $0 \leq r \leq n-1$, we have*

$$a_n \geq a_{n-1} \geq \cdots \geq a_r.$$

Then all the zeros of $p(q)$ lie in

$$|q| \leq \frac{1}{|a_n|}\{(|a_0| - a_r + a_n) + L\},$$

where

$$L = \sum_{l=1}^{r} |\alpha_l - \alpha_{l-1}|.$$

Remark 2. For
$$\alpha_r \leq \alpha_{r-1} \leq \cdots \leq \alpha_0.$$
Theorem 7 reduces to Theorem 6. For if we set $r = 0$, Theorem 7 gives the following corollary which is a significant improvements of some results in [7-9].

Corollary 7. *Let $p(q) = \sum_{l=0}^{n} q^l a_l$ be a polynomial of degree n with quaternion coefficients a and quaternion variable q, where $a_l = \alpha_l + i\beta_l + j\gamma_l + k\delta_l$ for $0 \leq l \leq n$. Suppose that we have*
$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_0.$$
Then all the zeros of $p(q)$ lie in
$$|q| \leq \frac{1}{|a_n|}\{(|\alpha_0| - \alpha_r + \alpha_n) + |\beta_0| + |\gamma_0| + |\delta_0| + M\},$$
where
$$M = \sum_{l=1}^{n}(|\beta_l - \beta_{l-1}| + |\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|).$$

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