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LIE IDEALS WITH GENERALIZED DERIVATIONS AND DERIVATIONS OF SEMIPRIME RINGS

Abstract. Let $\mathfrak{R}$ be a $2-$ torsion free semiprime ring, $\mathfrak{L}$ a square-closed Lie ideal of $\mathfrak{R}$, ϕ be a derivation of $\mathfrak{R}$ and α be an automorphism of $\mathfrak{R}$. We will demonstrate in this study that $\phi(\mathfrak{L}) = (0)$, and so ϕ is a zero map on $\mathfrak{L}$ if any one of the following holds for all $\mathfrak{r} \in \mathfrak{L}$: (i) $\phi(\mathfrak{r})\mathfrak{r} = 0$ (or $\mathfrak{r}\phi(\mathfrak{r}) = 0$) (ii) $\phi(\mathfrak{r})\mathfrak{r} + \mathfrak{r}(\alpha(\mathfrak{r}) - \mathfrak{r}) = 0$, (iii) The mapping $\mathfrak{r} \to \phi(\mathfrak{r}) + \alpha(\mathfrak{r})$ is commuting on $\mathfrak{L}$. Moreover, if any one of the following are satisfied for two generalized derivations $(\mathfrak{F}, \phi)$ and $(\mathfrak{H}, \xi)$ of $\mathfrak{R}$, then ϕ is a commuting map on $\mathfrak{L}$: (iv) $\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}) - \mathfrak{H}(\mathfrak{r}\mathfrak{s}) \in Z(R)$, (v) $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) = \pm\mathfrak{H}(\mathfrak{r}\mathfrak{s})$, (vi) $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) = \pm\mathfrak{H}(\mathfrak{s}\mathfrak{r})$, for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$.

Key words: semiprime ring, Lie ideal, derivation, generalized derivation.

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1. Introduction

$\mathfrak{R}$ will exhibit an associative ring with centre $Z(\mathfrak{R})$ throughout this article. The notation $[\mathfrak{r}, \mathfrak{s}]$ denotes the commutator $\mathfrak{r}\mathfrak{s} - \mathfrak{s}\mathfrak{r}$ for every $\mathfrak{r}, \mathfrak{s} \in \mathfrak{R}$, while the symbol $\mathfrak{r} \circ \mathfrak{s}$ denotes the anti-commutator $\mathfrak{r}\mathfrak{s} + \mathfrak{s}\mathfrak{r}$. Remember that a ring $\mathfrak{R}$ is prime if $\mathfrak{r}\mathfrak{R}\mathfrak{s} = \{0\}$ implies $\mathfrak{r} = 0$ or $\mathfrak{s} = 0$, and $\mathfrak{R}$ is semiprime if $\mathfrak{r}\mathfrak{R}\mathfrak{r} = \{0\}$ implies $\mathfrak{r} = 0$. An additive subgroup $\mathfrak{L}$ of $\mathfrak{R}$ is said to be a Lie ideal of $\mathfrak{R}$ if $[\mathfrak{r}, r] \in \mathfrak{L}$, for all $\mathfrak{r} \in \mathfrak{L}, r \in \mathfrak{R}$. An additive mapping $\phi : \mathfrak{R} \to \mathfrak{R}$ is called a derivation if $\phi(\mathfrak{r}\mathfrak{s}) = \phi(\mathfrak{r})\mathfrak{s} + \mathfrak{r}\phi(\mathfrak{s})$ holds for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{R}$.

Posner [13] established the commutativity of prime rings with derivation. The history of commuting and centralizing mappings dates to 1955 when Divinsky [9] established that if a simple Artinian ring has a nontrivial commuting automorphism, it is commutative. Posner has shown in [13] that if a prime ring has a nontrivial derivation that is centralising on the entire ring, it must be commutative. The results of Divinsky, which we just described, was generalized by Luh [12] to arbitrary prime rings. Mayne [11] established that if a prime ring has a nontrivial centralizing automorphism, the ring is commutative. A map $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$ is a generalized derivation of a ring $\mathfrak{R}$ associated with a derivation ϕ if $\mathfrak{F}$ is additive and satisfies $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) = \mathfrak{F}(\mathfrak{r})\mathfrak{s} + \mathfrak{r}\phi(\mathfrak{s}), \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{R}$. Derivations and generalized inner derivations (i.e., $\mathfrak{r} \to a\mathfrak{r} + \mathfrak{r}b$ for some $a, b \in \mathfrak{R}$) are basic examples. It's worth noting that the concept of generalized derivations encompasses both derivations and left multipliers (i.e., $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) = \mathfrak{F}(\mathfrak{r})\mathfrak{s}$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{R}$). As a result, it should be interesting to relate some of these results to generalized derivations.

Ashraf and Rehman demonstrated in [2] that $\mathfrak{R}$ is commutative if $\mathfrak{R}$ is a prime ring with nonzero ideal and ϕ is a derivation such that $\phi(\mathfrak{r}\mathfrak{s}) \pm \mathfrak{r}\mathfrak{s} \in Z(\mathfrak{R})$, for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{I}$. In [1] and [14], this theorem was considered for generalized derivations. All of these situations related with a square closed Lie ideal $\mathfrak{L}$ in a prime ring $\mathfrak{R}$ were studied in [10] Gölbaşi and Koç. These conditions have been generalized and discussed by Dhara et al. in the prime ring in [8].

We will extend the aforementioned conclusions for a nonzero Lie ideal of semiprime rings using derivation and generalized derivation of $\mathfrak{R}$ in the current article.

2. Preliminaries

We mention the following results which are crucial in developing the proof of our main result.

Lemma 1 ([4] Lemma 4). *Let $\mathfrak{R}$ be a prime ring with characteristic not two, $a, b \in \mathfrak{R}$. If $\mathfrak{L}$ a noncentral Lie ideal of $\mathfrak{R}$ and $a\mathfrak{L}b = 0$, then $a = 0$ or $b = 0$.*

Lemma 2 ([4] Lemma 5). *Let $\mathfrak{R}$ be a prime ring with characteristic not two and $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$. If ϕ is a nonzero derivation of $\mathfrak{R}$ such that $\phi(\mathfrak{L}) = (0)$, then $\mathfrak{L} \subseteq Z(\mathfrak{R})$.*

Lemma 3 ([4] Lemma 2). *Let $\mathfrak{R}$ be a prime ring with characteristic not two. If $\mathfrak{L}$ a noncentral Lie ideal of $\mathfrak{R}$, then $C_{\mathfrak{R}}(\mathfrak{L}) = Z(\mathfrak{R})$.*

Lemma 4 ([3] Theorem 7). *Let $\mathfrak{R}$ be a prime ring with characteristic not two and $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$. If ϕ is a nonzero derivation of $\mathfrak{R}$ such that $[\mathfrak{r}, \phi(\mathfrak{r})] \in Z(\mathfrak{R})$, for all $\mathfrak{r} \in \mathfrak{L}$, then $\mathfrak{L} \subseteq Z(\mathfrak{R})$.*

Lemma 5 ([15] Lemma 2.4). *Let $\mathfrak{R}$ be a 2-torsion free semiprime ring, $\mathfrak{L}$ is a Lie ideal of $\mathfrak{R}$ such that $\mathfrak{L} \not\subseteq Z(\mathfrak{R})$ and $a \in \mathfrak{L}$. If $a\mathfrak{L}a = 0$, then $a^2 = 0$ and there exists a nonzero ideal $K = \mathfrak{R}[\mathfrak{L}, \mathfrak{L}]\mathfrak{R}$ of $\mathfrak{R}$ generated by $[\mathfrak{L}, \mathfrak{L}]$ such that $[K, \mathfrak{R}] \subseteq \mathfrak{L}$ and $Ka = aK = 0$.*

Lemma 6 ([16] Corollary 2.1). *Let $\mathfrak{R}$ be a 2-torsion free semiprime ring, $\mathfrak{L}$ a noncentral Lie ideal of $\mathfrak{R}$ and $a, b \in \mathfrak{L}$.*

(*i*) *If* $a\mathfrak{L}a = 0$*, then* $a = 0$.
(*ii*) *If* $a\mathfrak{L} = 0$*(or* $\mathfrak{L}a = 0$*), then* $a = 0$
(*iii*) *If* $\mathfrak{L}$ *is square-closed and* $a\mathfrak{L}b = 0$*, then* $ab = 0$ *and* $ba = 0$.

Firstly, we prove the following lemma.

Lemma 7. *Let* $\mathfrak{R}$ *be a 2-torsion free semiprime ring and* $\mathfrak{L}$ *be a square closed Lie ideal of* $\mathfrak{R}$*. Suppose that the relation* $a\mathfrak{r}b + b\mathfrak{r}c = 0$ *holds some* $a, b, c \in \mathfrak{L}$ *and for all* $\mathfrak{r} \in \mathfrak{L}$ *. In this case* $(a + c)\,\mathfrak{r}b = 0$ *for all* $\mathfrak{r} \in \mathfrak{L}$.

Proof. By the hypothesis, we get

$$a\mathfrak{r}b + b\mathfrak{r}c = 0, \forall \mathfrak{r} \in \mathfrak{L}. \tag{1}$$

Replacing $\mathfrak{r}$ by $2\mathfrak{r}\mathfrak{s}$, we get $2a\mathfrak{r}\mathfrak{s}b + 2b\mathfrak{r}\mathfrak{s}c = 0$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$. Again replacing $\mathfrak{s}$ by $2b\mathfrak{s}$ and using the fact that $\mathfrak{R}$ is 2-torsion free, we get

$$a\mathfrak{r}b\mathfrak{s}b + b\mathfrak{r}b\mathfrak{s}c = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{2}$$

On the other hand right multiplication by $\mathfrak{s}b$ of (1) gives

$$a\mathfrak{r}b\mathfrak{s}b + b\mathfrak{r}c\mathfrak{s}b = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{3}$$

Subtracting (3) from (2) we obtain

$$b\mathfrak{r}\,(b\mathfrak{s}c - c\mathfrak{s}b) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{4}$$

The substitution $2\mathfrak{w}\mathfrak{r}$ for $\mathfrak{r}$ in (4) gives that $2b\mathfrak{w}\mathfrak{r}(b\mathfrak{s}c - c\mathfrak{s}b) = 0$. Now, replacing $\mathfrak{w}$ by $2\mathfrak{s}c$ and $\mathfrak{R}$ is 2-torsion free, we find that

$$b\mathfrak{s}c\mathfrak{r}\,(b\mathfrak{s}c - c\mathfrak{s}b) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{5}$$

Left multiplication by $c\mathfrak{s}$ of (4) gives

$$c\mathfrak{s}b\mathfrak{r}\,(b\mathfrak{s}c - c\mathfrak{s}b) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{6}$$

Subtracting (6) from (5) we obtain

$$(b\mathfrak{s}c - c\mathfrak{s}b)\,\mathfrak{r}\,(b\mathfrak{s}c - c\mathfrak{s}b) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{7}$$

By Lemma 6, we see

$$b\mathfrak{s}c - c\mathfrak{s}b = 0,$$

and so $b\mathfrak{s}c = c\mathfrak{s}b$. Using this relation (1), we have $(a + c)\,\mathfrak{r}b = 0$, $\mathfrak{r} \in \mathfrak{L}$. The proof of the lemma is complete. ∎

Lemma 8. *Let $\mathfrak{R}$ be a 2−torsion free semiprime ring, $\mathfrak{L}$ a Lie ideal of $\mathfrak{R}$, and let $\mathfrak{F}$ be a generalized derivation of $\mathfrak{R}$ associated with a nonzero derivation ϕ such that $\phi(\mathfrak{L}) \subseteq \mathfrak{L}$. If $\mathfrak{F}(rs) = 0$ for all $r, s \in$, then $\phi = 0$ on $\mathfrak{L}$ and $\mathfrak{L} \subseteq Z(\mathfrak{R})$. Moreover, if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$, then $\mathfrak{F} = 0$ on $\mathfrak{L}$.*

Proof. $0 = \mathfrak{F}(rst) = \mathfrak{F}(rs)t + rs\phi(t) = rs\phi(t)$ for all $r, s, t \in \mathfrak{L}$. So, we have $\phi(t)s\phi(t) = 0$ for all $s, t \in \mathfrak{L}$. Thus, $\phi(t) = 0$ for all $t \in \mathfrak{L}$ by Lemma 6(i). So, we get $\mathfrak{L} \subseteq Z(\mathfrak{R})$ by Lemma 2.

While, $0 = \mathfrak{F}(rs) = \mathfrak{F}(r)s$ for all $r, s \in \mathfrak{L}$. So, we have $\mathfrak{F}(r)s\mathfrak{F}(r) = 0$. If $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$, we get $\mathfrak{F}(r) = 0$ for all $r \in \mathfrak{L}$ by Lemma 6(i). ■

3. Derivations on Lie ideals in semiprime rings

Throughout the study, since $\mathfrak{R}$ is 2−torsion free ring, $\mathfrak{r}\mathfrak{s}$ will be written instead of $2\mathfrak{r}\mathfrak{s}$ for each $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$ in order to facilitate the equations.

Theorem 1. *Let $\mathfrak{R}$ be a 2−torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\Theta : \mathfrak{R} \to \mathfrak{R}$ be an additive mapping such that $\Theta(\mathfrak{L}) \subseteq \mathfrak{L}$. If $\Theta(\mathfrak{r})\mathfrak{r} = 0$ (or $\mathfrak{r}\Theta(\mathfrak{r}) = 0$), for all $\mathfrak{r} \in \mathfrak{L}$, then $\Theta(\mathfrak{L}) = (0)$, and so Θ is zero map on $\mathfrak{L}$.*

Proof. Assume that

$$\Theta(\mathfrak{r})\mathfrak{r} = 0, \forall \mathfrak{r} \in \mathfrak{L}. \tag{8}$$

The linearization of the above relation gives

$$\Theta(\mathfrak{r})\mathfrak{s} + \Theta(\mathfrak{s})\mathfrak{r} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{9}$$

Replacing $\mathfrak{s}$ by $\mathfrak{s}^2$ in the above relation gives

$$\Theta(\mathfrak{r})\mathfrak{s}^2 + \Theta\left(\mathfrak{s}^2\right)\mathfrak{r} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{10}$$

Right multiplication of (9) by $\mathfrak{s}$, we find

$$\Theta(\mathfrak{r})\mathfrak{s}^2 + \Theta(\mathfrak{s})\mathfrak{r}\mathfrak{s} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{11}$$

Subtracting (11) from (10), we obtain

$$\Theta\left(\mathfrak{s}^2\right)\mathfrak{r} - \Theta(\mathfrak{s})\mathfrak{r}\mathfrak{s} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{12}$$

Taking $\mathfrak{r}$ by $\mathfrak{r}\Theta(\mathfrak{s})$ in the last equation and using equation (8), we get

$$\Theta\left(\mathfrak{s}^2\right)\mathfrak{r}\Theta(\mathfrak{s}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Right multiplication of (12) by $\Theta(\mathfrak{s})$ and using the above relation, we see that

$$\Theta(\mathfrak{s})\,\mathfrak{r}\mathfrak{s}\Theta(\mathfrak{s}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{13}$$

Left multiplication of the relation (13) by $\mathfrak{s}$ gives

$$\mathfrak{s}\Theta(\mathfrak{s})\,\mathfrak{r}\mathfrak{s}\Theta(\mathfrak{s}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

By Lemma 6, we obtain

$$\mathfrak{s}\Theta(\mathfrak{s}) = 0, \forall \mathfrak{s} \in \mathfrak{L}.$$

Right multiplication of the relation (9) by $\Theta(\mathfrak{s})$ and using the above relation, we have

$$\Theta(\mathfrak{s})\,\mathfrak{r}\Theta(\mathfrak{s}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

By Lemma 6, we conclude that $\Theta(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \mathfrak{L}$. Hence $\Theta(\mathfrak{L}) = (0)$. That is, Θ is zero map on $\mathfrak{L}$. The proof of the theorem is complete. ■

Corollary 1. *Let $\mathfrak{R}$ be a 2−torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\phi : \mathfrak{R} \to \mathfrak{R}$ be a derivation such that $\phi(\mathfrak{L}) \subseteq \mathfrak{L}$. If $\phi(\mathfrak{r})\,\mathfrak{r} = 0$ (or $\mathfrak{r}\phi(\mathfrak{r}) = 0$), for all $\mathfrak{r} \in \mathfrak{L}$, then $\phi(\mathfrak{L}) = (0)$, and so ϕ is zero map on $\mathfrak{L}$.*

Theorem 2. *Let $\mathfrak{R}$ be a 2−torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\phi : \mathfrak{R} \to \mathfrak{R}$ be a derivation and α an automorphism of $\mathfrak{R}$ such that $\alpha(\mathfrak{L}) \subseteq \mathfrak{L}$. If the mapping $\mathfrak{r} \to \phi(\mathfrak{r}) + \alpha(\mathfrak{r})$ is commuting on $\mathfrak{L}$, then ϕ is commuting map on $\mathfrak{L}$.*

Proof. By the hypothesis, we have

$$[\phi(\mathfrak{r}) + \alpha(\mathfrak{r}), \mathfrak{r}] = 0, \forall \mathfrak{r} \in \mathfrak{L}. \tag{14}$$

The linearization of the relation, we get

$$[\phi(\mathfrak{r}) + \alpha(\mathfrak{r}), \mathfrak{s}] + [\phi(\mathfrak{s}) + \alpha(\mathfrak{s}), \mathfrak{r}] = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{15}$$

Putting $\mathfrak{s}$ by $\mathfrak{s}\mathfrak{r}$ in the last equation and using equation (14), we obtain

$$\begin{aligned} 0 &= [\phi(\mathfrak{r}) + \alpha(\mathfrak{r}), \mathfrak{s}]\,\mathfrak{r} + [\phi(\mathfrak{s}), \mathfrak{r}]\,\mathfrak{r} + [\mathfrak{s}, \mathfrak{r}]\,\phi(\mathfrak{r}) \\ &\quad + \mathfrak{s}\,[\phi(\mathfrak{r}), \mathfrak{r}] + [\alpha(\mathfrak{s}), \mathfrak{r}]\,\alpha(\mathfrak{r}) + \alpha(\mathfrak{s})\,[\alpha(\mathfrak{r}), \mathfrak{r}]. \end{aligned}$$

Using (14), replacing $\mathfrak{s}\,[\phi(\mathfrak{r}), \mathfrak{r}]$ by $-\mathfrak{s}\,[\alpha(\mathfrak{r}), \mathfrak{r}]$ in this equation, we have

$$\begin{aligned} 0 &= [\phi(\mathfrak{r}) + \alpha(\mathfrak{r}), \mathfrak{s}]\,\mathfrak{r} + [\phi(\mathfrak{s}), \mathfrak{r}]\,\mathfrak{r} + [\mathfrak{s}, \mathfrak{r}]\,\phi(\mathfrak{r}) \\ &\quad - \mathfrak{s}\,[\alpha(\mathfrak{r}), \mathfrak{r}] + [\alpha(\mathfrak{s}), \mathfrak{r}]\,\alpha(\mathfrak{r}) + \alpha(\mathfrak{s})\,[\alpha(\mathfrak{r}), \mathfrak{r}]. \end{aligned}$$

Also, using (15), replacing $[\phi(\mathfrak{r}) + \alpha(\mathfrak{r}), \mathfrak{s}]\mathfrak{r} + [\phi(\mathfrak{s}), \mathfrak{r}]\mathfrak{r}$ by $-[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{r}$ in the last equation, we get

$$0 = -[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{r} - \mathfrak{s}[\alpha(\mathfrak{r}), \mathfrak{r}] + [\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}) + [\alpha(\mathfrak{s}), \mathfrak{r}]\alpha(\mathfrak{r}) + \alpha(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}].$$

That is,

$$(16) \qquad [\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] + [\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}) = 0, \text{ for all } \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

where $\mathfrak{G}(\mathfrak{r})$ denotes $\alpha(\mathfrak{r}) - \mathfrak{r}$. Replacing $\mathfrak{s}$ by $\mathfrak{r}\mathfrak{s}$ in the last equation, we obtain

$$(17) \qquad \begin{aligned} 0 = \; & [\alpha(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \alpha(\mathfrak{r})[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] \\ & + \mathfrak{r}\mathfrak{G}(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] + \mathfrak{r}[\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}). \end{aligned}$$

Multiplying the relation (16) from the left side by $\mathfrak{r}$, we get

$$\mathfrak{r}[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{r}\mathfrak{G}(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] + \mathfrak{r}[\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}) = 0,$$

and so, subtracting (16) from (17), we have

$$\begin{aligned} 0 &= [\alpha(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \alpha(\mathfrak{r})[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] + \mathfrak{r}\mathfrak{G}(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] \\ &\quad + \mathfrak{r}[\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}) - \mathfrak{r}[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) - \mathfrak{r}\mathfrak{G}(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] - \mathfrak{r}[\mathfrak{s}, \mathfrak{r}]\phi(\mathfrak{r}) \\ &= [\alpha(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + (\alpha(\mathfrak{r}) - \mathfrak{r})[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})[\alpha(\mathfrak{r}), \mathfrak{r}] \\ &= [\alpha(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})[\mathfrak{G}(\mathfrak{r}), \mathfrak{r}] \end{aligned}$$

Using $[\alpha(\mathfrak{r}), \mathfrak{r}] = [\mathfrak{G}(\mathfrak{r}), \mathfrak{r}]$ in the above expression, we find that

$$0 = [\mathfrak{G}(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})[\alpha(\mathfrak{s}), \mathfrak{r}]\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})[\mathfrak{G}(\mathfrak{r}), \mathfrak{r}].$$

This implies that

$$\begin{aligned} 0 = \; & \mathfrak{G}(\mathfrak{r})\mathfrak{r}\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) - \mathfrak{r}\mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{r}\mathfrak{G}(\mathfrak{r}) \\ & - \mathfrak{G}(\mathfrak{r})\mathfrak{r}\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r})\mathfrak{r} - \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{r}\mathfrak{G}(\mathfrak{r}). \end{aligned}$$

and so,

$$\mathfrak{r}\mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})(-\mathfrak{G}(\mathfrak{r})\mathfrak{r}) = 0, \text{ for all } \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

By Lemma 7, we get

$$[\mathfrak{G}(\mathfrak{r}), \mathfrak{r}]\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) = 0, \text{ for all } \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Since α is an automorphism, we get

$$(18) \qquad \alpha^{-1}([\mathfrak{G}(\mathfrak{r}), \mathfrak{r}])\mathfrak{s}\alpha^{-1}(\mathfrak{G}(\mathfrak{r})) = 0, \text{ for all } \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Replacing $\mathfrak{s}\alpha^{-1}(\mathfrak{r})$ for $\mathfrak{s}$ in the above relation, we obtain

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])\,\mathfrak{s}\alpha^{-1}(\mathfrak{r})\,\alpha^{-1}(\mathfrak{G}(\mathfrak{r}))=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}.$$

Using α automorphism, we get

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])\,\mathfrak{s}\alpha^{-1}(\mathfrak{r}\mathfrak{G}(\mathfrak{r}))=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}. \tag{19}$$

Right multiplication of the relation (18) by $\alpha^{-1}(\mathfrak{r})$ gives,

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])\,\mathfrak{s}\alpha^{-1}(\mathfrak{G}(\mathfrak{r}))\,\alpha^{-1}(\mathfrak{r})=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}.$$

Since α is an automorphism, we obtain

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])\,\mathfrak{s}\alpha^{-1}(\mathfrak{G}(\mathfrak{r})\,\mathfrak{r})=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}. \tag{20}$$

Subtracting (19) from (20), we obtain

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])\,\mathfrak{s}\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}.$$

By Lemma 6, we get

$$\alpha^{-1}([\mathfrak{G}(\mathfrak{r}),\mathfrak{r}])=0,\forall\mathfrak{r},\mathfrak{s}\in\mathfrak{L}.$$

We conclude that $[\mathfrak{G}(\mathfrak{r}),\mathfrak{r}]=0$, for all $\mathfrak{r}\in\mathfrak{L}$. That is, $\mathfrak{G}$ is commuting map on $\mathfrak{L}$. Using $[\mathfrak{G}(\mathfrak{r}),\mathfrak{r}]=0$, we get

$$[\alpha(\mathfrak{r}),\mathfrak{r}]=0,\forall\mathfrak{r}\in\mathfrak{L}.$$

By the hypothesis, we get ϕ is commuting map on $\mathfrak{L}$. The proof of the theorem is complete. ■

Corollary 2. *Let $\mathfrak{R}$ be a be a prime ring with characteristic not two, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\phi:\mathfrak{R}\to\mathfrak{R}$ be a derivation and α an automorphism of $\mathfrak{R}$ such that $\alpha(\mathfrak{L})\subseteq\mathfrak{L}$. If the mapping $\mathfrak{r}\to\phi(\mathfrak{r})+\alpha(\mathfrak{r})$ is commuting on $\mathfrak{L}$, then $\mathfrak{L}\subseteq Z(\mathfrak{R})$.*

Proof. Using the same methods in the proof of Theorem 2, we have ϕ is commuting map on $\mathfrak{L}$. By Lemma 2, we get $\mathfrak{L}\subseteq Z(\mathfrak{R})$. ■

Theorem 3. *Let $\mathfrak{R}$ be a $2-$torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\phi:\mathfrak{R}\to\mathfrak{R}$ be a derivation and α an automorphism of $\mathfrak{R}$ such that $\alpha(\mathfrak{L})\subseteq\mathfrak{L}$. If $\phi(\mathfrak{r})\mathfrak{r}+\mathfrak{r}(\alpha(\mathfrak{r})-\mathfrak{r})=0$ for all $\mathfrak{r}\in\mathfrak{L}$, then $\phi(\mathfrak{L})=(0)$, and so ϕ is a zero map on $\mathfrak{L}$.*

Proof. We have

$$\phi(\mathfrak{r})\mathfrak{r} + \mathfrak{r}\mathfrak{G}(\mathfrak{r}) = 0, \forall \mathfrak{r} \in \mathfrak{L} \tag{21}$$

where $\mathfrak{G}(\mathfrak{r})$ stands for $\alpha(\mathfrak{r}) - \mathfrak{r}$. The linerazation of the last relation gives

$$\phi(\mathfrak{r})\mathfrak{s} + \phi(\mathfrak{s})\mathfrak{r} + \mathfrak{r}\mathfrak{G}(\mathfrak{s}) + \mathfrak{s}\mathfrak{G}(\mathfrak{r}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{22}$$

Replacing $\mathfrak{s}\mathfrak{r}$ for $\mathfrak{s}$ in the last equation and using (21), we obtain

$$\phi(\mathfrak{r})\mathfrak{s}\mathfrak{r} + \phi(\mathfrak{s})\mathfrak{r}^2 + \mathfrak{r}\mathfrak{G}(\mathfrak{s})\alpha(\mathfrak{r}) + \mathfrak{r}\mathfrak{s}\mathfrak{G}(\mathfrak{r}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{23}$$

Right multiplication of the relation (22) by $\mathfrak{r}$ gives

$$\phi(\mathfrak{r})\mathfrak{s}\mathfrak{r} + \phi(\mathfrak{s})\mathfrak{r}^2 + \mathfrak{r}\mathfrak{G}(\mathfrak{s})\mathfrak{r} + \mathfrak{s}\mathfrak{G}(\mathfrak{r})\mathfrak{r} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Subtracting the above relation from the relation (23), we obtain

$$\mathfrak{r}\mathfrak{G}(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{r}\mathfrak{s}\mathfrak{G}(\mathfrak{r}) - \mathfrak{s}\mathfrak{G}(\mathfrak{r})\mathfrak{r} = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{24}$$

Taking $\mathfrak{r}\mathfrak{s}$ for $\mathfrak{s}$ in the above relation and using (24) we obtain

$$\mathfrak{r}\mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{r}^2\mathfrak{G}(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) + \mathfrak{r}^2\mathfrak{s}\mathfrak{G}(\mathfrak{r}) - \mathfrak{r}\mathfrak{s}\mathfrak{G}(\mathfrak{r})\mathfrak{r} = 0,$$

and so

$$\mathfrak{r}\mathfrak{G}(\mathfrak{r})\alpha(\mathfrak{s})\mathfrak{G}(\mathfrak{r}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Since α is an automorphism, we have

$$\alpha^{-1}(\mathfrak{r}\mathfrak{G}(\mathfrak{r}))\mathfrak{s}\alpha^{-1}(\mathfrak{G}(\mathfrak{r})) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Writing $\mathfrak{s}\alpha^{-1}(\mathfrak{r})$ for $\mathfrak{s}$ in the last equation, we obtain

$$\alpha^{-1}(\mathfrak{r}\mathfrak{G}(\mathfrak{r}))\mathfrak{s}\alpha^{-1}(\mathfrak{r})\alpha^{-1}(\mathfrak{G}(\mathfrak{r})) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}.$$

Using α automorphism and Lemma 6, we get

$$\alpha^{-1}(\mathfrak{r}\mathfrak{G}(\mathfrak{r})) = 0, \forall \mathfrak{r} \in \mathfrak{L}.$$

That is, $\mathfrak{r}\mathfrak{G}(\mathfrak{r}) = 0$, $\mathfrak{r} \in \mathfrak{L}$. By Theorem 1, we have $\mathfrak{G}(\mathfrak{L}) = (0)$. Hence, we get $\phi(\mathfrak{r})\mathfrak{r} = 0$ by the hypothesis, and so $\phi(\mathfrak{L}) = (0)$ by Corollary 1. The proof of the theorem is complete. ■

Corollary 3. *Let $\mathfrak{R}$ be a prime ring with characteristic not two, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\phi : \mathfrak{R} \to \mathfrak{R}$ be a derivation and α an automorphism of $\mathfrak{R}$ $\alpha(\mathfrak{L}) \subseteq \mathfrak{L}$. If $\phi(\mathfrak{r})\mathfrak{r} + \mathfrak{r}(\alpha(\mathfrak{r}) - \mathfrak{r}) = 0$ for all $\mathfrak{r} \in \mathfrak{L}$. then $\mathfrak{L} \subseteq Z(\mathfrak{R})$.*

Proof. By the same techniques in the proof of Theorem 3, we get $\phi(\mathfrak{L}) = (0)$. By Lemma 2, we conclude that $\mathfrak{L} \subseteq Z(\mathfrak{R})$. ■

4. Generalized derivations on Lie ideals in semiprime rings

Theorem 4. *Let $\mathfrak{R}$ be a $2-$torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and $\mathfrak{F}$, $\mathfrak{H}$ generalized derivations associated with the derivations ϕ, ξ of $\mathfrak{R}$ respectively such that $\phi(\mathfrak{L}) \subseteq \mathfrak{L}$. If $\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}) \pm \mathfrak{H}(\mathfrak{r}\mathfrak{s}) \in Z(\mathfrak{R})$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$, then ϕ is commuting on $\mathfrak{L}$.*

Proof. By the hypothesis, we have

$$\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}) - \mathfrak{H}(\mathfrak{r}\mathfrak{s}) \in Z(\mathfrak{R}), \text{ for all } \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{25}$$

In the above relation, replacing $\mathfrak{s}$ by $\mathfrak{s}\mathfrak{w}$ for $\mathfrak{w} \in \mathfrak{L}$, we have

$$\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}\mathfrak{w}) - \mathfrak{H}(\mathfrak{r}\mathfrak{s}\mathfrak{w}) \in Z(\mathfrak{R}), \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L},$$

which gives

$$(\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}) - \mathfrak{H}(\mathfrak{r}\mathfrak{s}))\mathfrak{w} + \mathfrak{F}(\mathfrak{r})\mathfrak{s}\phi(\mathfrak{w}) - \mathfrak{r}\mathfrak{s}\xi(\mathfrak{w}) \in Z(\mathfrak{R}), \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}.$$

Commuting with $\mathfrak{w}$, we have

$$[(\mathfrak{F}(\mathfrak{r})\mathfrak{F}(\mathfrak{s}) - \mathfrak{H}(\mathfrak{r}\mathfrak{s}))\mathfrak{w}, \mathfrak{w}] + [\mathfrak{F}(\mathfrak{r})\mathfrak{s}\phi(\mathfrak{w}) - \mathfrak{r}\mathfrak{s}\xi(\mathfrak{w}), \mathfrak{w}] = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}.$$

Using (25), we obtain

$$[\mathfrak{F}(\mathfrak{r})\mathfrak{s}\phi(\mathfrak{w}) - \mathfrak{r}\mathfrak{s}\xi(\mathfrak{w}), \mathfrak{w}] = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}. \tag{26}$$

Now replacing $\mathfrak{r}$ by $\mathfrak{r}\mathfrak{y}, \mathfrak{y} \in \mathfrak{L}$ in (26), we get

$$[(\mathfrak{F}(\mathfrak{r})\mathfrak{y} + \mathfrak{r}\phi(\mathfrak{y}))\mathfrak{s}\phi(\mathfrak{w}) - \mathfrak{r}\mathfrak{y}\mathfrak{s}\xi(\mathfrak{w}), \mathfrak{w}] = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}. \tag{27}$$

Taking $\mathfrak{s}$ by $\mathfrak{y}\mathfrak{s}$ in equation (26), we have

$$[\mathfrak{F}(\mathfrak{r})\mathfrak{y}\mathfrak{s}\phi(\mathfrak{w}) - \mathfrak{r}\mathfrak{y}\mathfrak{s}\xi(\mathfrak{w}), \mathfrak{w}] = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}. \tag{28}$$

Subtracting (28) from (27), we arrive at

$$[\mathfrak{r}\phi(\mathfrak{y})\mathfrak{s}\phi(\mathfrak{w}), \mathfrak{w}] = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}. \tag{29}$$

Replacing $\mathfrak{r}$ by $\mathfrak{t}\mathfrak{r}$, $t \in \mathfrak{L}$ and using (29), above relation gives

$$[\mathfrak{t}, \mathfrak{w}]\mathfrak{r}\phi(\mathfrak{y})\mathfrak{s}\phi(\mathfrak{w}) = 0, \forall \mathfrak{s}, \mathfrak{w}, \mathfrak{y}, \mathfrak{t} \in \mathfrak{L}.$$

Replacing $\mathfrak{w}$ by $\mathfrak{y}$, above relation gives

$$[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})\,\mathfrak{s}\phi(\mathfrak{y})=0,\forall\mathfrak{r},\mathfrak{s},\mathfrak{y},\mathfrak{t}\in\mathfrak{L}.$$

Replacing $\mathfrak{s}$ by $\mathfrak{s}[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}$ in the above equation, we get

$$[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})\,\mathfrak{s}[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})=0,\forall\mathfrak{r},\mathfrak{s},\mathfrak{y},\mathfrak{t}\in\mathfrak{L}.$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})=0,\text{for all }\mathfrak{r},\mathfrak{y},t\in\mathfrak{L}.$$

Replacing $\mathfrak{t}$ by $\phi(\mathfrak{y})$ in the above equation, we get

$$[\phi(\mathfrak{y}),\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})=0,\forall\mathfrak{r},\mathfrak{y}\in\mathfrak{L}. \tag{30}$$

Multiplying (30) on the right by $\mathfrak{y}$, we get

$$[\phi(\mathfrak{y}),\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})\,\mathfrak{y}=0,\forall\mathfrak{r},\mathfrak{y}\in\mathfrak{L}. \tag{31}$$

Taking $\mathfrak{r}$ by $\mathfrak{r}\mathfrak{y}$ in equation (30), we have

$$[\phi(\mathfrak{y}),\mathfrak{y}]\,\mathfrak{r}\mathfrak{y}\phi(\mathfrak{y})=0,\forall\mathfrak{r},\mathfrak{y}\in\mathfrak{L}. \tag{32}$$

Subtracting (31) from (32), we have

$$[\phi(\mathfrak{y}),\mathfrak{y}]\,\mathfrak{r}\,[\phi(\mathfrak{y}),\mathfrak{y}]=0,\forall\mathfrak{r},\mathfrak{y}\in\mathfrak{L}.$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\phi(\mathfrak{y}),\mathfrak{y}]=0,\ \text{for all }\mathfrak{y}\in\mathfrak{L}.$$

which gives ϕ is commuting map on $\mathfrak{L}$. In a similar manner, we can prove that the same conclusion holds for $\mathfrak{F}(\mathfrak{r})\,\mathfrak{F}(\mathfrak{s})+\mathfrak{H}(\mathfrak{r}\mathfrak{s})\in Z(\mathfrak{R})$ for all $\mathfrak{r},\mathfrak{s}\in\mathfrak{L}$. The proof of the theorem is complete. ■

Corollary 4. *Let $\mathfrak{R}$ be a prime ring with characteristic not two, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and $\mathfrak{F},\mathfrak{H}$ generalized derivations associated with the derivations ϕ,ξ of $\mathfrak{R}$ respectively. If $\mathfrak{F}(\mathfrak{r})\,\mathfrak{F}(\mathfrak{s})\pm\mathfrak{H}(\mathfrak{r}\mathfrak{s})\in Z(\mathfrak{R})$ for all $\mathfrak{r},\mathfrak{s}\in\mathfrak{L}$, then $\mathfrak{L}\subseteq Z(\mathfrak{R})$.*

Proof. Using the same procedures as used in the proof of Theorem 4, we have

$$[\mathfrak{t},\mathfrak{y}]\,\mathfrak{r}\phi(\mathfrak{y})=0,\text{for all }\mathfrak{r},\mathfrak{y},\mathfrak{t}\in\mathfrak{L}. \tag{33}$$

By Lemma 1, either $[\mathfrak{t},\mathfrak{y}]=0$ or $\phi(\mathfrak{y})=0$, for each $\mathfrak{y}\in\mathfrak{L}$. Now, we set $\alpha=\{\mathfrak{y}\in\mathfrak{L}\mid[\mathfrak{t},\mathfrak{y}]=0,\forall\mathfrak{t}\in\mathfrak{L}\}$, $\beta=\{\mathfrak{y}\in\mathfrak{L}\mid\phi(\mathfrak{y})=0\}$, then α and β are additive subgroup of $\mathfrak{L}$ and $\mathfrak{L}=\alpha\cup\beta$. Since a group cannot be the union of its two proper subgroups, either $\alpha=\mathfrak{L}$ or $\beta=\mathfrak{L}$. If $\alpha=\mathfrak{L}$, then $\mathfrak{L}\subseteq Z(\mathfrak{R})$ by Lemma 3. On the other hand if $\beta=\mathfrak{L}$, then $\mathfrak{L}\subseteq Z$ by Lemma 2. ■

Theorem 5. *Let $\mathfrak{R}$ be a 2-torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\mathfrak{F}$, $\mathfrak{H}$ be generalized derivations associated with derivations ϕ, ξ of $\mathfrak{R}$ respectively such that $\phi(\mathfrak{L}) \subseteq \mathfrak{L}$ and $\xi(\mathfrak{L}) \subseteq \mathfrak{L}$.*

If $\mathfrak{F}(\mathfrak{rs}) = \pm\mathfrak{H}(\mathfrak{rs})$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$ and $\phi \mp \xi \neq 0$, then $\phi = \pm\xi$ on $\mathfrak{L}$ and $\mathfrak{L} \subseteq Z(\mathfrak{R})$.

Moreover, if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$ and $\mathfrak{H}(\mathfrak{L}) \subseteq \mathfrak{L}$, then $\mathfrak{F} = \pm\mathfrak{H}$ on $\mathfrak{L}$.

Proof. We set $H = \mathfrak{F} \pm \mathfrak{H}$ and $h = \phi \mp \xi$. By the hypothesis, $H(\mathfrak{rs}) = 0$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$. By Lemma 8, we have $h = 0$ on $\mathfrak{L}$ and $\mathfrak{L} \subseteq Z(\mathfrak{R})$. So, we have $\phi = \pm\xi$ on $\mathfrak{L}$.

Moreover, we assume that $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$ and $\mathfrak{H}(\mathfrak{L}) \subseteq \mathfrak{L}$. Then $H(\mathfrak{L}) \subseteq \mathfrak{L}$ implies that $H = 0$ on $\mathfrak{L}$ by Lemma 8, so we have $\mathfrak{F} = \pm\mathfrak{H}$ on $\mathfrak{L}$. ■

Corollary 5. *Let $\mathfrak{R}$ be a prime ring with characteristic not two, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and let $\mathfrak{F}$, $\mathfrak{H}$ be generalized derivations associated with derivations ϕ, ξ of $\mathfrak{R}$ respectively such that $\phi(\mathfrak{L}) \subseteq \mathfrak{L}$ and $\xi(\mathfrak{L}) \subseteq \mathfrak{L}$.*

If $\mathfrak{F}(\mathfrak{rs}) = \pm\mathfrak{H}(\mathfrak{rs})$ for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$ and $\phi \mp \xi \neq 0$, then $\phi = \pm\xi$ on $\mathfrak{L}$ and $\mathfrak{L} \subseteq Z(\mathfrak{R})$.

Moreover, if $\mathfrak{F}(\mathfrak{L}) \subseteq \mathfrak{L}$ and $\mathfrak{H}(\mathfrak{L}) \subseteq \mathfrak{L}$, then $\mathfrak{F} = \pm\mathfrak{H}$ on $\mathfrak{L}$.

Theorem 6. *Let $\mathfrak{R}$ be a 2-torsion free semiprime ring, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and $\mathfrak{F}, \mathfrak{H}$ generalized derivations associated with the derivations ϕ, ξ of $\mathfrak{R}$ respectively such that $\phi(\mathfrak{r}) \in \mathfrak{L}$, for all $\mathfrak{r} \in \mathfrak{L}$. If $\mathfrak{F}(\mathfrak{rs}) = \pm\mathfrak{H}(\mathfrak{sr})$, for all $\mathfrak{r}, \mathfrak{s} \in \mathfrak{L}$, then ϕ is commuting on $\mathfrak{L}$.*

Proof. Suppose that

$$\mathfrak{F}(\mathfrak{rs}) - \mathfrak{H}(\mathfrak{sr}) = 0, \forall \mathfrak{r}, \mathfrak{s} \in \mathfrak{L}. \tag{34}$$

Replacing $\mathfrak{s}$ by $\mathfrak{sw}$, $\mathfrak{w} \in \mathfrak{L}$ in the above equation, we get

$$\mathfrak{F}(\mathfrak{rs})\mathfrak{w} + \mathfrak{rs}\phi(\mathfrak{w}) - \mathfrak{H}(\mathfrak{sw})\mathfrak{r} - \mathfrak{sw}\xi(\mathfrak{r}) = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}.$$

This implies that

$$(\mathfrak{F}(\mathfrak{rs}) - \mathfrak{H}(\mathfrak{sr}))\mathfrak{w} + \mathfrak{H}(\mathfrak{sr})\mathfrak{w} + \mathfrak{rs}\phi(\mathfrak{w}) - \mathfrak{H}(\mathfrak{sw})\mathfrak{r} - \mathfrak{sw}\xi(\mathfrak{r}) = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}.$$

Using (34), we have

$$\mathfrak{H}(\mathfrak{s})\mathfrak{rw} + \mathfrak{s}\xi(\mathfrak{r})\mathfrak{w} + \mathfrak{rs}\phi(\mathfrak{w}) - \mathfrak{H}(\mathfrak{s})\mathfrak{wr} - \mathfrak{s}\xi(\mathfrak{w})\mathfrak{r} - \mathfrak{sw}\xi(\mathfrak{r}) = 0,$$

for all $\mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}$, and so

$$\mathfrak{H}(\mathfrak{s})[\mathfrak{r}, \mathfrak{w}] + \mathfrak{s}[\xi(\mathfrak{r}), \mathfrak{w}] + \mathfrak{rs}\phi(\mathfrak{w}) - \mathfrak{s}\xi(\mathfrak{w})\mathfrak{r} = 0, \forall \mathfrak{r}, \mathfrak{s}, \mathfrak{w} \in \mathfrak{L}.$$

Writing $\mathfrak{w}$ by $\mathfrak{r}$ in the last equation, we get

$$\begin{aligned} 0 &= \mathfrak{H}(\mathfrak{s})[\mathfrak{r},\mathfrak{r}] + \mathfrak{s}[\xi(\mathfrak{r}),\mathfrak{r}] + \mathfrak{r}\mathfrak{s}\phi(\mathfrak{r}) - \mathfrak{s}\xi(\mathfrak{r})\mathfrak{r} \\ &= \mathfrak{s}\xi(\mathfrak{r})\mathfrak{r} - \mathfrak{s}\mathfrak{r}\xi(\mathfrak{r}) + \mathfrak{r}\mathfrak{s}\phi(\mathfrak{r}) - \mathfrak{s}\xi(\mathfrak{r})\mathfrak{r}. \end{aligned}$$

We have

$$\mathfrak{r}\mathfrak{s}\phi(\mathfrak{r}) - \mathfrak{s}\mathfrak{r}\xi(\mathfrak{r}) = 0, \forall \mathfrak{r},\mathfrak{s} \in \mathfrak{L}.$$

Writing $\mathfrak{s}$ by $\mathfrak{s}\mathfrak{w}$ and using this equation, we have

$$\mathfrak{r}\mathfrak{s}\mathfrak{w}\phi(\mathfrak{r}) = \mathfrak{s}\mathfrak{w}\mathfrak{r}\xi(\mathfrak{r}) = \mathfrak{s}\mathfrak{r}\mathfrak{w}\phi(\mathfrak{r}), \forall \mathfrak{r},\mathfrak{s} \in \mathfrak{L},$$

this implies that

$$[\mathfrak{r},\mathfrak{s}]\mathfrak{w}\phi(\mathfrak{r}) = 0, \forall \mathfrak{r},\mathfrak{s} \in \mathfrak{L}.$$

Writing $\mathfrak{s}$ by $\phi(\mathfrak{r})$, we have

$$[\mathfrak{r},\phi(\mathfrak{r})]\mathfrak{w}\phi(\mathfrak{r}) = 0, \forall \mathfrak{r},\mathfrak{w} \in \mathfrak{L}. \tag{35}$$

Multiplying (35) on the right by $\mathfrak{r}$, we get

$$[\mathfrak{r},\phi(\mathfrak{r})]\mathfrak{w}\phi(\mathfrak{r})\mathfrak{r} = 0, \forall \mathfrak{r},\mathfrak{w} \in \mathfrak{L}. \tag{36}$$

Taking $\mathfrak{w}$ by $\mathfrak{w}\mathfrak{r}$ in equation (35), we have

$$[\mathfrak{r},\phi(\mathfrak{r})]\mathfrak{w}\mathfrak{r}\phi(\mathfrak{r}) = 0, \forall \mathfrak{r},\mathfrak{w} \in \mathfrak{L}. \tag{37}$$

Subtracting (36) from (37), we have

$$[\mathfrak{r},\phi(\mathfrak{r})]\mathfrak{w}[\mathfrak{r},\phi(\mathfrak{r})] = 0, \forall \mathfrak{r},\mathfrak{w} \in \mathfrak{L}.$$

By Lemma 6, $[\mathfrak{r},\phi(\mathfrak{r})] = 0$, for all $\mathfrak{r} \in \mathfrak{L}$. Hence, ϕ is commuting on $\mathfrak{L}$.

In a similar manner, we can prove that the same conclusion holds for $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) + \mathfrak{H}(\mathfrak{s}\mathfrak{r}) = 0$, for all $\mathfrak{r},\mathfrak{s} \in \mathfrak{L}$. The proof of the theorem is complete. ■

Corollary 6. *Let $\mathfrak{R}$ be a prime ring with characteristic not two, $\mathfrak{L}$ a square closed Lie ideal of $\mathfrak{R}$ and $\mathfrak{F},\mathfrak{H}$ generalized derivations associated with the derivations ϕ,ξ of $\mathfrak{R}$ respectively. If $\mathfrak{F}(\mathfrak{r}\mathfrak{s}) = \pm\mathfrak{H}(\mathfrak{s}\mathfrak{r})$, for all $\mathfrak{r},\mathfrak{s} \in \mathfrak{L}$, then $\mathfrak{L} \subseteq Z(\mathfrak{R})$.*

Proof. Using the same methods in the proof of Theorem 6, we have $[\mathfrak{r},\mathfrak{s}]\mathfrak{w}\phi(\mathfrak{r}) = 0$, for all $\mathfrak{r},\mathfrak{s},\mathfrak{w} \in \mathfrak{L}$. This equation is the same as the equation (33). Using the same methods in the proof of Corollary 4, we get $\mathfrak{L} \subseteq Z(\mathfrak{R})$. ■

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References

[1] Ashraf M., Ali A., Ali S., Some commutativity theorems for rings with generalized derivations, *Southeast Asian Bull. Math.,* 31(2007), 415-421.

[2] Ashraf M., Rehman N., On derivations and commutativity in prime rings, *East-West J. Math.,* 3(1)(2001), 87-91.

[3] Awtar R., Lie structure in prime rings with derivations, *Publ. Math. Debrecen,* 31(1984), 209-215.

[4] Bergen J., Herstein I.N., Kerr W., Lie ideals and derivation of prime rings, *J. of Algebra* 71*,* (1981), 259-267.

[5] Bresar M., On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math. J.* 33*,* (1991), 89-93.

[6] Bresar M., Centralizing mappings and derivations in prime rings, *J. Algebra 156,* (1993), 385-394.

[7] Bresar M., On skew-commuting mappings of rings, *Bull. Austral. Math. Soc. 47,* (1993), 291-296.

[8] Dhara B., Rehman N., Raza M.A., Lie ideals and action of generalized derivations in rings, *Miskolc Math Notes,* 16(2)(2015), 769-779.

[9] Divinsky N., On commuting automorphisms of rings, *Trans. Roy. Soc. Canada Sect. III. 49,* (1955), 19-52.

[10] Gölbaşı Ö., Koç E., Notes on commutativity of prime rings with generalized derivations, *Commun. Fac. Sci. Ank. Series A1 58,* (2009), 39-46.

[11] Mayne J.H., Centralizing automorphisms of prime rings, *Canad. Math. Bull. 19,* (1976), 113-115.

[12] Luh J., A note on commuting automorphisms of rings, *Amer. Math. Monthly 77,* (1970), 61-62.

[13] Posner E.C., Derivations in prime rings, *Proc. Amer. Math. Soc. 8,* (1957), 1093-1100.

[14] Quadri M.A., Khan M.S., Rehman N., Generalized derivations and commutativity of prime rings, *Indian J. Pure Appl. Math.,* 34(2003), 1393-1396.

[15] Rehman N., Hongan M., Generalized Jordan derivations on Lie ideals associate with Hochschild 2-cocycles of rings, *Rend. Circ. Mat. Palermo,* 60(3)(2011), 437-444.

[16] Rehman N., Hongan M., Al-Omary R.M., Lie ideals and Jordan triple derivations in rings, *Rend. Sem. Mat. Univ. Padova,* 125(2011), 147-156.

[17] Vukman J., Identities with derivations and autommorphisms on semiprime rings, *Internat J. Math. and Math. Sci.,* 7(2005), 1031-1038.

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