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Ç. A. Bektaş, A. Karakaş and Y. Altin

THE KÖTHE-TOEPLITZ DUALS OF SOME GENERALIZED $p-$CONVEX SEQUENCE SPACES

Abstract. In this paper, we define $\alpha-, \beta-$ and $\gamma-$ duals of the sequence spaces $\Delta_p^m(Z)$ for $Z = \ell_\infty, c$ and c_0. We study on some matrix transformations of these sequence spaces.

Key words: sequence spaces, dual space, matrix transformations.

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1. Introduction

Throughout this study, we denote the space of all complex sequences by w and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $z = (z_k)$ with complex terms, respectively normed by $\|z\|_\infty = \sup_k |z_k|$, where $k \in \mathbb{N}$.

In 1981, the forward difference sequence spaces $\Delta(Z)$ were introduced by Kızmaz [16]. He showed that $Z \subset \Delta(Z)$, where Z is any ℓ_∞, c or c_0. For instance, if we take $(z_k) = (k)$, $(k = 1, 2, 3, ...)$, then the sequence (z_k) is not convergent but it is $\Delta-$convergent. He also studied their topological properties, $\alpha-, \beta-, \gamma-$duals of these spaces. Later, in 1995, Et and Çolak [10] defined the forward generalized difference sequence spaces $\Delta^m(Z)$.

The notion of backward difference sequence spaces was generalized by Malkowsky and Parashar [22]. Let m be a non-negative integer. Then,

$$\Delta^m(Z) = \{z = (z_k) : (\Delta^m z_k) \in Z\}$$
$$\Delta^0 z = (z_k), \Delta^m z = \left(\Delta^{m-1} z_{k+1} - \Delta^{m-1} z_k\right)$$

and so

$$\Delta^m z_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} z_{k-i}.$$

The sequence spaces $\Delta^m(Z)$ are Banach spaces normed by

$$\|z\|_\Delta = \sum_{i=1}^{m} |z_i| + \|\Delta^m z_k\|_\infty .$$

Out of these, using the generalized difference operator $\mathbf{\Delta}^m$, Ioan [13] introduced the concept of $p-$convex sequences as the following:

Let K be the set of all real sequences and $p \in \mathbb{R}\setminus\{0\}$. Then the linear operator $\Delta_p^m : K \to K$, $m \in \mathbb{N}$ is defined such that

$$(\Delta_p z_k) = (z_{k+1} - p z_k), \quad \left(\Delta_p^{m+1} z_k\right) = \Delta_p(\Delta_p^m z_k) = \left(\Delta_p^m z_{k+1} - p\Delta_p^m z_k\right)$$

and

$$\Delta_p^m z = \left(\Delta_p^m z_k\right) = \sum_{v=0}^{m} (-1)^{m-v} \binom{m}{v} p^{m-v} z_{k+v}.$$

Hence we define the sequence spaces $\Delta_p^m(Z) = \left\{z = (z_k) : \left(\Delta_p^m z_k\right) \in Z\right\}$ for $Z = \ell_\infty, c$ or c_0.

Furthermore a sequence (z_k) from K is said to be $p-$convex of order $m \in \mathbb{N}$ if and only if $\Delta_p^m z_k \geq 0$, for all $k \in \mathbb{N}$. Later on, Karakaş *et al.* [15] defined and studied some basic topological and algebraic properties of the sequence spaces $\Delta_p^m(Z)$ for $Z = \ell_\infty, c$ and c_0 where $p, m \in \mathbb{N}$. We study on the sets of sequences, which are Δ_p^m-bounded, Δ_p^m-convergent and Δ_p^m-zero. The sequence space $\Delta^m(Z)$ is different from the sequence space $\Delta_p^m(Z)$ and $\Delta^m(Z) \cap \Delta_p^m(Z) \neq \emptyset$ (for $Z = \ell_\infty, c$ and c_0). Recently the difference sequence spaces have been studied by many researchers Altin [2], Braha [3], Et and Nuray [11], Et, et al. [12], Tripathy [23].

l_1, cs, bv, bv_0 and bs are defined by Kamthan and Gupta [14] as the following

$$\ell_1 = \left\{ z = (z_k) : \sum_{k=1}^{\infty} |z_k| < \infty \right\},$$

$$cs = \left\{ z = (z_k) : \sum_{k=1}^{\infty} z_k \text{ is convergent} \right\},$$

$$bv = \left\{ z = (z_k) : \sum_{k=1}^{\infty} |z_{k+1} - z_k| < \infty \right\},$$

$$bv_0 = \left\{ z = (z_k) : z \in bv \text{ such that } \lim_{k\to\infty} z_k = 0 \right\},$$

$$bs = \left\{ z = (z_k) : \sup_n \left| \sum_{k=1}^{n} z_k \right| < \infty \right\}.$$

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [18], whose main results concerned $\alpha-$duals. An account of duals of sequence spaces is found in Köthe [17]. One can find about different types of duals of sequence spaces in Cooke [7], Çolak and Et [8] Kamthan and Gupta [14], Maddox [20], and many others.

Let Z be a sequence space and define

$$Z^{\alpha}=\left\{b=(b_k):\sum_{k=1}^{\infty}|b_k z_k|<\infty,\ \text{for all } z\in Z\right\},$$

$$Z^{\beta}=\left\{b=(b_k):\sum_{k=1}^{\infty}b_k z_k \text{ is convergent for all } z\in Z\right\},$$

$$Z^{\gamma}=\left\{b=(b_k):\ \sup_n\left|\sum_{k=1}^{n}b_k z_k\right|<\infty \text{ for all } z\in Z\right\}, \text{see [14].}$$

Then Z^{α}, Z^{β} and Z^{γ} are called $\alpha-,\beta-$ and $\gamma-$ duals of Z, respectively. It is clear that $Z^{\alpha}\subset Z^{\beta}\subset Z^{\gamma}$ for $Z=\ell_{\infty}, c$ or c_0.

If $Z\subset Y$, then $Y^{\eta}\subset Z^{\eta}$, for $\eta=\alpha,\beta$ or γ. We shall write $Z^{\eta\eta}=(Z^{\eta})^{\eta}$ for $\eta=\alpha,\beta$ or γ.

Ahmad and Mursaleen [1], Başarır [4],Bektaş *et al.* [5], Chandra and Tripathy [6], Et [9], Lascarides [19], Maddox [21] and others have studied results involving $\alpha-$and $\beta-$duals of different sequence spaces and their properties.

2. Main results

In this section, we give $\alpha-,\beta-$ and $\gamma-$ duals of $\Delta_p^m(Z)$, for $Z=\ell_{\infty}, c$ or c_0.

Theorem 1. *Let Z be ℓ_{∞}, c or c_0 and $m\in\mathbb{N}$, $p\in\mathbb{R}\backslash\{0\}$. Then*
i) $\left[\Delta_p^m(Z)\right]^{\alpha}=\ell_1$,
ii) $\left[\Delta_p^m(Z)\right]^{\alpha\alpha}=\ell_{\infty}$.

Proof. *i)* Suppose that $b\in\ell_1$. Then

$$\sum_{k=1}^{\infty}|b_k|<\infty. \tag{1}$$

Let $z\in\Delta_p^m(Z)$. Then there is a positive integer M such that $\left|\Delta_p^m z_k\right|\leq M$, $(k=1,2,3,...)$. We also write

$$z_k=(-1)^m p^{-m}\Delta_p^m z_k\sum_{v=1}^{m}(-1)^{m+v}p^{-m+v-1}\Delta_p^{m-v}z_{k+1}.$$

Then

$$\sum_{k=1}^{\infty}|b_k z_k|=\sum_{k=1}^{\infty}|b_k|\left|(-1)^m p^{-m}\Delta_p^m z_k+\sum_{v=1}^{m}(-1)^{m+v}p^{-m+v-1}\Delta_p^{m-v}z_{k+1}\right|$$
$$\leq M\left|p^{-m}\right|\sum_{k=1}^{\infty}|b_k|+M\sum_{k=1}^{\infty}|b_k|\left|\sum_{v=1}^{m}(-1)^{m+v}p^{-m+v-1}\right|<\infty.$$

Thus $\ell_1 \subset \left[\Delta_p^m(Z)\right]^\alpha$.

Conversely suppose that $b \in \left[\Delta_p^m(c_0)\right]^\alpha$ and $b \notin \ell_1$. Then there exists $m \in \mathbb{N}$ such that

$$\sum_{k=1}^{m} |b_k| = \infty.$$

Define $z \in \Delta_p^m(c_0)$ by

$$z_k = \begin{cases} 0, & k > m \\ 1, & k \leq m \end{cases}.$$

Then we have

$$\sum_{k=1}^{\infty} |b_k z_k| = \sum_{k=1}^{m} |b_k z_k| + \sum_{k=m+1}^{\infty} |b_k z_k|$$
$$= \sum_{k=1}^{m} |b_k| = \infty.$$

This contradicts to $b \in \left[\Delta_p^m(c_0)\right]^\alpha$. Hence $b \in \ell_1$. This complete the proof of i).

ii) Since $\left[\Delta_p^m(Z)\right]^\alpha = \ell_1$, we have $\left[\Delta_p^m(Z)\right]^{\alpha\alpha} = \ell_1^\alpha = \ell_\infty$. ■

Theorem 2. *Let Z be ℓ_∞, c or c_0 and $m \in \mathbb{N}$ and $p \in \mathbb{R}\setminus\{0\}$. Then*
i) $\left[\Delta_p^m(Z)\right]^\beta = cs$,
ii) $\left[\Delta_p^m(Z)\right]^{\beta\beta} = bv$.

Proof. We will proof for $Z = \ell_\infty$. It can be shown for $Z = c$ or c_0.

i) Let $b \in cs$ and $z \in \Delta_p^m(\ell_\infty)$. Then the series $\sum_{k=1}^{\infty} b_k$ is convergent and since $z \in \Delta_p^m(\ell_\infty)$, there exists a positive integer M such that $\left|\Delta_p^m z_k\right| \leq M$. Then we may write

$$\sum_{k=1}^{\infty} b_k z_k = \sum_{k=1}^{\infty} b_k \left[(-1)^m p^{-m} \Delta_p^m z_k + \sum_{v=1}^{m} (-1)^{m+v} p^{-m+v-1} \Delta_p^{m-v} z_{k+1}\right].$$

Hence $\sum_{k=1}^{\infty} b_k z_k$ is convergent for all $z \in \Delta_p^m(\ell_\infty)$, so $b \in \left[\Delta_p^m(\ell_\infty)\right]^\beta$. Now let $b \in \left[\Delta_p^m(\ell_\infty)\right]^\beta \setminus cs$. Then $\sum_{k=1}^{\infty} b_k$ is divergent, that is $\sum_{k=1}^{\infty} b_k = \infty$. We define the sequence $z = (z_k)$ by $z_k = 1$ for all $k \in \mathbb{N}$.

Then $z \in \Delta_p^m(\ell_\infty)$ and we may write

$$\sum_{k=1}^{\infty} b_k z_k = \sum_{k=1}^{\infty} b_k = \infty.$$

This contradicts to $b \in \left[\Delta_p^m(\ell_\infty)\right]^\beta$. Hence $b \in cs$.
ii) Since $\left[\Delta_p^m(Z)\right]^\beta = cs$, we have $\left[\Delta_p^m(Z)\right]^{\beta\beta} = cs^\beta = bv$. ■

Theorem 3. *Let Z be ℓ_∞, c or c_0, $m \in \mathbb{N}$ and $p \in \mathbb{R}\setminus\{0\}$. Then:*
i) $\left[\Delta_p^m(Z)\right]^\gamma = bs$,
ii) $\left[\Delta_p^m(Z)\right]^{\gamma\gamma} = bv$.

Proof. *i*) and *ii*) can be proved by the same way as Theorem 2. ■

3. Matrix transformations

Given any infinite matrix $B = (b_{nk})_{n,k=1}^\infty$ of complex numbers and any sequence $z = (z_k)$, we write

$$B_n(z) = \sum_{k=1}^\infty b_{nk} z_k, (n = 1, 2, ...)$$

and $Bx = (B_n(z))_{n=1}^\infty$, provided the series $\sum_{k=1}^\infty b_{nk} z_k$ are convergent for each $n \in \mathbb{N}$.

Theorem 4. *Let $G = l_\infty, c$ and $H = l_\infty, c$. Then $B = (b_{nk}) \in \left(\Delta_p^m(G), H\right)$ if and only if $\left(\sum_k |b_{nk}|\right) \in H$.*

Proof. Let G and H be ℓ_∞.

Necessity. Let $B \in \left(\Delta_p^m(\ell_\infty), \ell_\infty\right)$. Then $B_n(z) = \sum_{k=1}^\infty b_{nk} z_k$ is convergent for each $n \in \mathbb{N}$ and $(B_n(z)) \in \ell_\infty$ for all $z \in \Delta_p^m(\ell_\infty)$. If we take $z = (z_k)$ by $z_k = sgn b_{nk}$ we have $z \in \Delta_p^m(\ell_\infty)$ and

$$\begin{aligned} \sup_n |B_n(z)| &= \sup_n \left| \sum_k b_{nk} z_k \right| \\ &= \sup_n \sum_k |b_{nk}| < \infty. \end{aligned}$$

Sufficiency. Let $z \in \Delta_p^m(\ell_\infty)$ and $\sup_n \sum_k |b_{nk}| < \infty$. Then we obtain that

$$\sup_n \left| \sum_k b_{nk} z_k \right| \leq \sup_n \sum_k |b_{nk}| \, |z_k| \leq K \sup_n \sum_k |b_{nk}| < \infty.$$

Hence $B \in \left(\Delta_p^m(\ell_\infty), \ell_\infty\right)$.
The proof can be given easily for the other cases. ■

Theorem 5. *Let $G = \ell_\infty, c$ and $H = \ell_\infty, c$ or c_0. Then $B = (b_{nk}) \in \left(G, \Delta_p^m(H)\right)$ if and only if*

i) $\sum\limits_{k=1}^{\infty} |b_{nk}| < \infty$ *for each n,*

ii) $C \in (G, H)$,

where $C = (c_{nk}) = \left(\Delta_p^{m-1} b_{n+1,k} - p\Delta_p^{m-1} b_{nk}\right)$.

Proof. Let G and H be ℓ_∞.

Necessity. Let $B \in \left(\ell_\infty, \Delta_p^m(\ell_\infty)\right)$. Then $B_n(z) = \sum\limits_k b_{nk} z_k$ is convergent for $z \in \ell_\infty$ and $(B_n(z)) \in \Delta_p^m(\ell_\infty)$. Since $B_n(z)$ converges, we have

$$B_n(z) = \left| \sum_k b_{nk} z_k \right| = \sum_k |b_{nk}| \,.$$

If we choose $z_k = sgn b_{nk}$, then we obtain that $\sup\limits_n \sum\limits_k |b_{nk}| < \infty$ for each n. Thus $i)$ holds.

Since $(B_n(z)) \in \Delta_p^m(\ell_\infty)$ for $z \in \ell_\infty$, we have

$$\begin{aligned}
\left(\Delta_p^m(B_n(z))\right) &= \left(\Delta_p^m \left(\sum_k b_{nk} z_k\right)\right) \\
&= \left(\sum_k \Delta_p^m b_{nk} z_k\right) \\
&= \left(\sum_k (\Delta_p^{m-1} b_{n+1,k} - p\Delta_p^{m-1} b_{nk}) z_k\right) \in \ell_\infty.
\end{aligned}$$

If we take $C = (c_{nk}) = \left(\Delta_p^{m-1} b_{n+1,k} - p\Delta_p^{m-1} b_{nk}\right)$ from (1), we have

$$\begin{aligned}
(C_n(z)) &= \left(\sum_k c_{nk} z_k\right) \\
&= \left(\sum_k (\Delta_p^{m-1} b_{n+1,k} - p\Delta_p^{m-1} b_{nk}) z_k\right)
\end{aligned}$$

and $(C_n(z)) \in \ell_\infty$. Thus $ii)$ holds.

Sufficiency. Suppose that $i)$ and $ii)$ hold. Let $z \in \ell_\infty$. Then from $i)$, we obtain

$$|B_n(z)| = \left| \sum_k b_{nk} z_k \right| \le \sup_k |z_k| \sum_k |b_{nk}| < \infty.$$

Also from ii), we get

$$\left(\Delta_p^m\left(\sum_k b_{nk}z_k\right)\right)=\left(\sum_k(\Delta_p^{m-1}b_{n+1,k}-p\Delta_p^{m-1}b_{nk})z_k\right)\in\ell_\infty.$$

The proof can be shown for the other cases. ■

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ÇIĞDEM A. BEKTAŞ
DEPARTMENT OF MATHEMATICS
FIRAT UNIVERSITY
ELAZIG - TURKEY
e-mail: cbektas@firat.edu.tr

ABDULKADIR KARAKAŞ
DEPARTMENT OF MATHEMATICS
SIIRT UNIVERSITY
SIIRT -TURKEY
e-mail: kadirkarakas21@hotmail.com

YAVUZ ALTIN
DEPARTMENT OF MATHEMATICS
FIRAT UNIVERSITY
ELAZIG - TURKEY
e-mail: yaltin23@yahoo.com

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