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THE VORONOVSKAYA TYPE THEOREM FOR THE BASKAKOV-KANTOROVICH OPERATORS

ABSTRACT: In this paper we give the Voronovskaya type theorem for the Baskakov-Kantorovich operators in the polynomial weighted spaces.

The Baskakov operators were studied in [1]. The Voronovskaya type theorem for some operators are given in [2], [3].

KEY WORDS: Voronovskaya theorem, Baskakov-Kantorovich operator, weighted space.

1. PRELIMINARY

We take the notation like in M. Becker's paper [1], i.e.: $N := \{1,2,....\}$, $N_0 := N \cup \{0\}$, $R_0 := [0,+\infty)$ and let for a fixed $p \in N_0$ and for all $x \in R_0$

$$(1) \qquad w_0(x) \equiv 1, \quad w_p(x) = \frac{1}{1+x^p} \quad \text{if} \quad p \geq 1.$$

Let $C_p = C_p(R_0)$ being the set of all real - valued functions continuous on R_0, for which $w_p(\cdot)f(\cdot)$ is uniformly continuous and bounded on R_0. The norm in C_p is defined by the formula

$$(2) \qquad \| f \|_{C_p} := \sup_{x \in R_0} w_p(x) |f(x)|.$$

Let $m \in N$, $p \in N_0$, being fixed numbers. Denote by C_p^m, the set of $f \in C_p$, which $f^{(k)}$, $k = 0,1,...,m$, belong to C_p.

In paper [1] were studied the Baskakov operators for functions $f \in C_p$

$$(3) \qquad V_n(f;x) = \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right), \qquad x \in R_0, \; n \in N,$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k} \quad \text{for} \quad k \in N_0.$$

Now we introduce the Baskakov-Kantrovich operators T_n for $f \in C_p$, $p \in N_0$,

$$T_n(f;x)=\sum_{k=0}^{\infty} b_{n,k}(x)n\int_{k/n}^{(k+1)/n} f(t)dt, \qquad x\in R_0,\ n\in N. \tag{4}$$

From (4) it follows that

$$T_n(1;x)=1 \qquad \text{for} \qquad x\in R_0,\ n\in N. \tag{5}$$

The futher properties T_n we shall give in Section 2. The Voronovskaya type theorem will be given in Section 3.

2. Lemmas

In this part we will need Becker's results of the Baskakov operators V_n defined by the formula (3).

Lemma 1 ([1]). *Let $p\in N_0$ be a fixed number. Then there exists a positive constant $M_3(p)$ depending only on p such that for all $n\in N$ and $x\in R_0$ we have*

$$w_p(x)V_n\left(\frac{(t-x)^2}{w_p(x)};x\right)\le M_3(p)\frac{x(1+x)}{n}. \tag{6}$$

Now we shall give some properties of the operators T_n.

Lemma 2. For all $n\in N$ and $x\in R_0$ we have

$$T_n(t-x;x)=\frac{1}{2n},$$

$$T_n((t-x)^2;x)=\frac{x^2}{n}+\frac{x}{n}+\frac{1}{3n^2},$$

$$T_n((t-x)^4;x)=3\frac{x^4}{n^2}+6\frac{x^4}{n^3}+6\frac{x^3}{n^2}+16\frac{x^3}{n^3}+3\frac{x^2}{n^2}+15\frac{x^2}{n^3}+5\frac{x}{n^3}+\frac{1}{5n^4}. \tag{7}$$

Proof. For example we shall prove (7). From the linearity of the operator T_n we get

$$T_n((t-x)^4;x)=T_n(t^4;x)+4xT_n(t^3;x)+6x^2T_n(t^2;x)-4x^3T_n(t;x)+x^4T_n(1;x).$$

From (5) we know, that $T_n(1;x)=1$. Using the definition of the operators T_n we can calculate, that for $x\in R_0$, $n\in N$, we have

$$T_n(t;x) = x + \frac{1}{2n},$$

$$T_n(t^2;x) = \frac{(n+1)}{n}x^2 + 2\frac{1}{n}x + \frac{1}{3n^2},$$

$$T_n(t^3;x) = \frac{(n+1)(n+2)}{n^2}x^3 + \frac{9}{2}\frac{(n+1)}{n^2}x^2 + \frac{7}{2}\frac{1}{n^2} + \frac{1}{4n^3},$$

$$T_n(t^4;x) = \frac{(n+1)(n+2)(n+3)}{n^3}x^4 + 8\frac{(n+1)(n+2)}{n^3}x^3 +$$

$$+15\frac{(n+1)}{n^3}x^2 + 6\frac{x}{n^3} + \frac{1}{5n^4},$$

which yield that

$$T_n((t-x)^4;x) = 3\frac{x^4}{n^2} + 6\frac{x^4}{n^3} + 6\frac{x^3}{n^2} + 16\frac{x^3}{n^3} + 3\frac{x^2}{n^2} + 15\frac{x^2}{n^3} + 5\frac{x}{n^3} + \frac{1}{5n^4}$$

for $x \in R_0$, $n \in N$.

Using the mathematical induction we can prove an analogous formula given in [1] for $V_n(t^q;x)$, $q \in N$.

Lemma 3. For $q \in N_0$, $n \in N$, $x \in R_0$ *there exist the positive coefficients* $\xi_{j,n,q}$, $0 \le j \le q$, *depending only on* q, j, n *and bounded for* n *and* x *such that*

$$T_n(t^q;x) = \sum_{j=0}^{q} \xi_{j,n,q}\, x^j\, n^{j-q}, \tag{8}$$

where $1 < \xi_{q,n,q} \le q!$.

Now we shall prove

Lemma 4. For every fixed $p \in N_0$ *there exists a positive constant* $M_1(p)$ *depending only on p such that for all* $n \in N$ *we have*

$$\left\| T_n\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{C_p} \le M_1(p). \tag{9}$$

Proof. By (1), (2) and (5) the inequality (9) is obvious if $p = 0$. Fix $p \in N$. From (1), (5) and the linearity of the operator T_n we get for $x \in R_0$ and $n \in N$

$$w_p(x)T_n\left(\frac{1}{w_p(t)};x\right)=w_p(x)T_n(1+t^p;x)\le$$

$$\le w_p(x)\left(1+p!x^p+\sum_{j=0}^{p-1}\xi_{j,n,p}x^j n^{j-p}\right)\le$$

$$\le p!+\sum_{j=0}^{p-1}\xi_{j,n,p}\frac{x^j}{1+x^p}n^{j-p}.$$

But $0\le x^j/(1+x^p)\le 1$ for $x\in R_0$ and $1\le j\le p-1$. Hence

$$w_p(x)T_n\left(\frac{1}{w_p(t)};x\right)\le p!+\frac{1}{n}\sum_{j=0}^{p-1}\xi_{j,n,p}n^{j-(p-1)}\le$$

$$\le p!+\frac{1}{n}\sum_{j=0}^{p-1}\xi_{j,n,p}\le M_1(p)=\text{const.}$$

for all $x\in R_0$ and $n\in N$, which implies

$$\left\|T_n\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{C_p}\le M_1(p)\qquad \text{for all}\quad n\in N.$$

Thus the proof of (9) is completed.

Applying Lemma 4 we shall prove

Lemma 5. For every fixed $p\in N_0$ there exists a positive constant $M_1(p)$ depending only on p such that any $f\in C_p$

(10) $$\|T_n(f;\cdot)\|_{C_p}\le M_1(p)\|f\|_{C_p},\qquad n\in N,$$

which proves that T_n, $n\in N$, is a linear positive operator from the space C_p into C_p.

Proof. From (2) we have

$$\|T_n(f;\cdot)\|_{C_p}=\sup_{x\in R_0}w_p(x)|T_n(f;x)|.$$

But from (4) we get

$$w_p(x)|T_n(f;x)| \le w_p(x)\sum_{k=0}^{\infty} b_{n,k}(x)x^k n \int_{k/n}^{(k+1)/n} |f(t)|w_p(t)\frac{1}{w_p(t)dt} \le$$

$$\le \|f\|_{C_p} w_p(x)T_n\left(\frac{1}{w_p(t)};x\right) \le \|f\|_{C_p} \left\| T_n\left(\frac{1}{w_p(t)};\cdot\right)\right\|_{C_p}.$$

Using Lemma 4, we immediately obtain (10).

Lemma 6. For some $x_0 \in R_0$ *there exists a positive constant* $M_2(x_0)$, *depending only on* x_0, *such that for all* $n \in N$ *we have*

(11) $$T_n((t-x_0)^4;x_0) \le M_2(x_0)n^{-2}.$$

Proof. Applying (7) we get

$$T_n((t-x_0)^4;x_0) = 3\frac{x_0^4}{n^2} + 6\frac{x_0^4}{n^3} + 6\frac{x_0^3}{n^2} + 16\frac{x_0^3}{n^3} + 3\frac{x_0^2}{n^2} + 15\frac{x_0^2}{n^3} + 4\frac{x_0}{n^3} + \frac{1}{5n^4} \le$$

$$\le (9x_0^4 + 22x_0^3 + 18x_0^2 + 5x_0 + 0{,}2)n^{-2} \equiv M_2(x_0)n^{-2}.$$

for every $n \in N$. Hence we have (11).

Lemma 7. For every $x_0 \in R_0$ *holds*

(12) $$\lim_{n\to\infty} nT_n(t-x_0;x_0) = \frac{1}{2},$$

(13) $$\lim_{n\to\infty} nT_n((t-x_0)^2;x_0) = x_0(1+x_0).$$

Proof. From Lemma 2 we know that

$$T_n(t-x_0;x_0) = \frac{1}{2n},$$

$$T_n((t-x_0)^2;x_0) = \frac{x_0^2}{n} + \frac{x_0}{n} + \frac{1}{3n^2},$$

for all $n \in N$ and $x_0 \in R_0$, which immediately yield (12) and (13).

Lemma 8. Let $x_0 \in R_0$ *be a fixed point and let* $g(\cdot;x_0)$ *be a given function belonging to* C_p *with some* $p \in N_0$ *and such that*

(14) $$\lim_{t\to x_0} g(t;x_0) = 0.$$

Then

(15) $$\lim_{t\to x_0} T_n(g(t;x_0);x_0)=0.$$

Proof. Choose $\varepsilon>0$ and a constant $M_1(p)$ like in Lemma 4. Then from (14) and properties of $g(\cdot;x_0)$ there exist the positive constants $\delta=\delta(\varepsilon,M_1)$ and M_4 such that

(16) $$w_p(t)\left|g(t;x_0)\right|<\frac{\varepsilon}{2M_1}\quad \text{for}\quad |t-x_0|<\delta,$$

(17) $$w_p(t)\left|g(t;x_0)\right|<\frac{\varepsilon}{2M_1}\quad \text{for all}\quad t\in R_0.$$

Notation by $Q_{n;1}:=\{k\in N_0:\left|k/n-x_0\right|<\delta\}$ and $Q_{n;1}:=\{k\in N_0:\left|k/n-x_0\right|\geq\delta\}$. Then for all $n\in N$ we have

$$w_p(x_0)\left|T_n(g(t;x_0);x_0)\right|\leq w_p(x_0)\sum_{k=0}^{\infty}b_{n,k}(x_0)n\left|\int_{k/n}^{(k+1)/n}g(t;x_0)dt\right|=$$

$$=w_p(x_0)\sum_{k\in Q_{n:1}}b_{n,k}(x_0)n\left|\int_{k/n}^{(k+1)/n}g(t;x_0)dt\right|+$$

$$+w_p(x_0)\sum_{k\in Q_{n:2}}b_{n,k}(x_0)n\left|\int_{k/n}^{(k+1)/n}g(t;x_0)dt\right|\equiv\Sigma_1+\Sigma_2.$$

From (16) and Lemma 4 we get

$$\Sigma_1\leq\frac{\varepsilon}{2M_1}w_p(x_0)\sum_{k=0}^{\infty}b_{n,k}(x_0)n\int_{k/n}^{(k+1)/n}\frac{1}{w_p(t)}dt\leq$$

$$\leq\frac{\varepsilon}{2M_1}w_p(x_0)T_n\left(\frac{1}{w_p(t)};x_0\right)\leq\frac{\varepsilon}{2M_1}M_1=\frac{\varepsilon}{2}.$$

Moreover by (17)

$$\Sigma_2\leq M_4w_p(x_0)\sum_{k\in Q_{n:2}}b_{n,k}(x_0)n\int_{k/n}^{(k+1)/n}\frac{1}{w_p(t)}dt.$$

But

$$\int_{k/n}^{(k+1)/n}\frac{1}{w_p(t)}dt\leq\frac{1}{n}\frac{1}{w_p((k+1)/n)}\leq\frac{2^{p+1}}{n}\frac{1}{w_p(k/n)}.$$

Thus

$$\Sigma_2 \le M_4 2^{p+1} w_p(x_0) \sum_{k \in Q_{n;2}} b_{n,k}(x_0) \frac{1}{w_p(k/n)}.$$

If $|k/n - x_0| \ge \delta$ then $1 \le \delta^{-2}(k/n - x_0)^2$. So

$$\Sigma_2 \le M_4 2^{p+1} \delta^{-2} w_p(x_0) \sum_{k \in Q_{n:2}} b_{n,k}(x_0) \left(\frac{k}{n} - x_0\right)^2 \frac{1}{w_p(k/n)} \le$$

$$\le M_4 2^{p+1} \delta^{-2} w_p(x_0) \sum_{k=0}^{\infty} b_{n,k}(x_0) \left(\frac{k}{n} - x_0\right)^2 \frac{1}{w_p(k/n)} \le$$

$$\le M_4 2^{p+1} \delta^{-2} w_p(x_0) V_n\left(\frac{(t - x_0)^2}{w_p(t)}; x_0\right).$$

Using Lemma 1 we get

$$\Sigma_2 \le M_4\, M_3(p) 2^{p+1} \frac{x_0(1 + x_0)}{n\delta^2} \equiv M_5(p) \frac{x_0(1 + x_0)}{n\delta^2}.$$

For the fixed positive numbers ε, δ, M_5 and $x_0 \ge 0$ there exists natural number n_1, depending only on ε, δ, M_5 such that for all $n > n_1$ we have

$$M_5(p) \frac{x_0(1 + x_0)}{n\delta^2} < \frac{\varepsilon}{2}.$$

Then $\Sigma_2 < \varepsilon/2$ for all $n > n_1$.

Consequently,

$$w_p(x_0)\left|T_n(g(t; x_0); x_0)\right| < \varepsilon \qquad \text{for all} \quad n > n_1,$$

which denotes

$$\lim_{n \to \infty} w_p(x_0)\, T_n(g(t; x_0); x_0) = 0.$$

From this and by the definition (1) we get (15).

3. The Voronovskaya type theorem

Now we shall prove the main theorem

Theorem. Let $f \in C_p^2$ *with some* $p \in N_0$*. Then for every* $x \in R_0$ *we have*

(18) $$\lim_{n\to\infty} n\{T_n(f;x)-f(x)\}=\frac{1}{2}f'(x)+\frac{x(1+x)}{2}f''(x).$$

Proof. We take a fixed point $x_0 \in R_0$. By the Taylor formula for $f \in C_p^2$ and $t \in R_0$ we have

$$f(t)=f(x_0)+f'(x_0)(t-x_0)+\frac{1}{2}f''(x_0)(t-x_0)^2+\varphi(t;x_0)(t-x_0)^2,$$

where $\varphi(\cdot;x_0)\in C_p$ and $\lim_{t\to x_0}\varphi(t;x_0)=0$. From this we derive

(19) $$\begin{aligned}T_n(f(t);x_0)&=f(x_0)T_n(1;x_0)+f'(x_0)T_n(t-x_0;x_0)+\\&+\frac{1}{2}f''(x_0)T_n((t-x_0)^2;x_0)+T_n(\varphi(t;x_0)(t-x_0)^2;x_0)\end{aligned}$$

for every $n \in N$. By Lemma 7 we get from (19)

(20) $$\begin{aligned}\lim_{n\to\infty} n\{T_n(f(t);x_0)-f(x_0)\}&=\frac{1}{2}f'(x_0)+\frac{x_0(x_0+1)}{2}f''(x_0)+\\&+\lim_{n\to\infty}T_n(\varphi(t;x_0)(t-x_0)^2;x_0).\end{aligned}$$

We shall prove that

(21) $$\lim_{n\to\infty} T_n(\varphi(t;x_0)(t-x_0)^2;x_0)=0.$$

Using the Hölder inequality we have

(22) $$\left|T_n(\varphi(t;x_0)(t-x_0)^2;x_0)\right|\le\left\{T_n(\varphi^2(t;x_0);x_0)\right\}^{1/2}\left\{T_n((t-x_0)^4;x_0)\right\}^{1/2}.$$

But by Lemma 6 we have

(23) $$T_n((t-x_0)^4;x_0)\le M_2(x_0)n^{-2} \qquad \text{for all} \quad n\ge 1.$$

Further the properties of function $\varphi(\cdot;x_0)$ let write that $\psi(t;x_0)\equiv\varphi^2(t;x_0)$ belongs to C_{2p} and $\lim_{t\to x_0}\psi(t;x_0)=0$. Hence we get by Lemma 8

(24) $$\lim_{n\to\infty} T_n(\psi(t;x_0);x_0)\equiv\lim_{n\to\infty} T_n(\varphi^2(t;x_0);x_0)=0.$$

Combining (22) – (24) we obtain (21), which using to (20) we get the desired assertion (18).

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