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APPROXIMATION THEOREMS FOR SOME OPERATORS IN WEIGHTED SPACES OF FUNCTIONS OF TWO VARIABLES

ABSTRACT: We study approximation properties of the Baskakov-Szasz-Mirakyan operators in some weighted spaces of functions of two variables. For these operators we give two theorems on the degree of approximation and the Voronovskaya type theorem.

The Szasz-Mirakyan and Baskakov operators in the polynomial and exponential weighted spaces of functions of one variable were examined in [1]-[4]. Approximation theorems and the Voronovskaya type theorem for some operators of the Szasz-Mirakyan type were given in [5]-[7].

KEY WORDS:Szasz-Mirakyan operator, Baskakov operator, weighted space, function of two variables.

1. PRELIMINARIES

1.1. We use the following notation $N:=\{1,2,...\}$, $N_0:=N\cup\{0\}$, $R_+:=$ $:=(0,+\infty)$, $R_0:=R_+\cup\{0\}$, $R_0^2:=R_0\times R_0$ and similarly as in [1] and [2] let for fixed $p\in N_0$, $q\in R_+$ and for all $x\in R_0$

(1) $$w_0(x):=1,\quad w_p(x):=(1+x^p)^{-1}\qquad \text{if}\quad p\geq 1,$$

(2) $$v_q(x):=e^{-qx}.$$

Next, for fixed $p\in N_0$ and $q\in R_+$, we introduce the weighted function

(3) $$w^*_{p,q}(x,y):=w_p(x)v_q(x),\qquad (x,y)\in R_0^2,$$

and the space $C_{p,q}$ of all real-valued functions f defined on R_0^2 such that $w_p(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2 and the norm is given by

(4) $$\|f\|_{p,q}\equiv\|f(\cdot,\cdot)\|_{p,q}:=\sup_{(x,y)\in R_0^2} w^*_{p,q}(x,y)|f(x,y)|.$$

For $f\in C_{p,q}$ we define the modulus of continuity

(5) $$\omega(f,C_{p,q};t,s):=\sup_{\substack{0\leq h\leq t\\ 0\leq\delta\leq s}}\|\Delta_{h,\delta}f(\cdot,\cdot)\|_{p,q},\ t,s\geq 0$$

where $\Delta_{h,\delta}f(x,y) = f(x+h,y+\delta) - f(x,y)$ for all $(x,y) \in R_0^2$, $h,\delta \in R_0$. Moreover, for fixed $m \in N$, $p \in N_0$, $q \in R_+$, let $C_{p,q}^m$ be the space of all functions $f \in C_{p,q}$ which the partial derivatives of the order $\leq m$ belong also to $C_{p,q}$.

1.2. In the papers [1, 2] were considered the Szasz-Mirakyan operators for functions f defined on R_0

$$S_n^{\{1\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{k}{n}\right), \tag{6}$$

$$S_n^{\{2\}}(f;x) := \sum_{k=0}^{\infty} a_{n,k}(x) n \int_{k/n}^{(k+1)/n} f(t)dt, \tag{7}$$

for $x \in R_0$ and $n \in N$, where

$$a_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}, \qquad k \in N_0, \tag{8}$$

and the Baskakov operators

$$B_n^{\{1\}}(f;x) := \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{9}$$

$$B_n^{\{2\}}(f;x) := \sum_{k=0}^{\infty} b_{n,k}(x) n \int_{k/n}^{(k+1)/n} f(t)dt, \tag{10}$$

for $x \in R_0$ and $n \in N$, where

$$b_{n,k}(x) := \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \qquad k \in N_0. \tag{11}$$

In our paper we shall consider the following operators $L_{m,n}^{\{i\}}$ for functions $f \in C_{p,q}$

$$L_{m,n}^{\{1\}}(f;x \quad y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x) a_{n,k}(y) f\left(\frac{j}{m}, \frac{k}{n}\right), \tag{12}$$

$$L_{m,n}^{\{2\}}(f;xy) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x) a_{n,k}(y) mn \int_{j/m}^{(j+1)/m} \int_{k/n}^{(k+1)/m} f(t,z)\, dt\, dz, \tag{13}$$

(14) $$L_{m,n}^{\{3\}}(f;xy) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)a_{n,k}(y)f\left(\frac{j}{m},\frac{k}{n}\right),$$

(15) $$L_{m,n}^{\{4\}}(f;xy) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)a_{n,k}(y)mn\int_{j/m}^{(j+1)/m}\int_{k/n}^{(k+1)/m} f(t,z)\,dt\,dz,$$

$(x,y)\in R_0^2$ and $m,n\in N$.

From (6) – (15) if follows that

(16) $$S_n^{\{i\}}(1;x) = 1 = B_n^{\{i\}}(1;x) \qquad \text{for } x\in R_0,\ n\in N,\ i=1,2,$$

(17) $$L_{m,n}^{\{i\}}(1;x,y) = 1 \qquad \text{for } (x,y)\in R_0^2,\ m,n\in N,\ 1\le i\le 4.$$

Moreover, if $f\in C_{p,q}$ with some $p\in N_0$ and $q\in R_+$, and $f(x,y)=f_1(x)f_2(y)$ for $(x,y)\in R_0^2$, then

(18) $$L_{m,n}^{\{i\}}(f;x,y) = S_m^{\{i\}}(f_1;x)S_n^{\{i\}}(f_2;y), \qquad i=1,2,$$

(19) $$L_{m,n}^{\{i\}}(f;x,y) = B_m^{\{i-2\}}(f_1;x)S_n^{\{i-2\}}(f_2;y), \qquad i=3,4,$$

for $m,n\in N$ and $(x,y)\in R_0^2$.

In Sec. 2 we shall give some basic properties of these operators. In Sec. 3 we shall prove the main theorems.

Below by $M_k(a,b)$, $k=1,2,\ldots$, we shall denote the suitable positive constants depending only on indicated parameters.

2. Auxiliary results

2.1. First we shall give some properties of the operators $S_n^{\{i\}}$ and $B_n^{\{i\}}$.

Lemma 1 ([1]). *For all $n\in N$, $x\in R_0$ we have*

$$S_n^{\{1\}}(t-x;x)=0, \qquad S_n^{\{2\}}(t-x;x)=S_n^{\{1\}}(t-x;x)+\frac{1}{2n},$$

$$S_n^{\{1\}}((t-x)^2;x)=\frac{x}{n}, \qquad S_n^{\{2\}}((t-x)^2;x)=S_n^{\{1\}}((t-x)^2;x)+\frac{1}{3n^2},$$

$$S_n^{\{1\}}((t-x)^3;x)=\frac{x}{n^2},$$

$$S_n^{\{1\}}((t-x)^4;x)=\frac{3x^2}{n^2}+\frac{x}{n^3},$$

$$S_n^{\{2\}}((t-x)^3;x) = S_n^{\{1\}}((t-x)^3;x) + \frac{3}{2n} S_n^{\{1\}}(t-x;x) +$$
$$+\frac{1}{n^2} S_n^{\{1\}}(t-x;x) + \frac{1}{4n^3} = \frac{5}{2}\frac{x}{n^2} + \frac{1}{4n^3},$$
$$S_n^{\{2\}}((t-x)^4;x) = S_n^{\{1\}}((t-x)^4;x) + \frac{2}{n} S_n^{\{1\}}((t-x)^3;x) +$$
$$+\frac{2}{n^2} S_n^{\{1\}}((t-x)^2;x) + \frac{1}{5n^4} = \frac{3x^2}{n^2} + \frac{5x}{n^3} + \frac{1}{5n^4},$$
$$B_n^{\{1\}}(t-x;x) = 0,$$
$$B_n^{\{1\}}((t-x)^2;x) = \frac{x(1+x)}{n},$$
$$B_n^{\{1\}}((t-x)^3;x) = \frac{x(1+x)(1+2x)}{n^2},$$
$$B_n^{\{1\}}((t-x)^4;x) = \frac{3(n+2)}{n^3}x^4 + \frac{6(n+2)}{n^3}x^3 + \frac{7n+3}{n^3}x^2 + \frac{x}{n^3},$$
$$B_n^{\{2\}}(t-x;x) = B_n^{\{1\}}(t-x;x) + \frac{1}{2n} = \frac{1}{2n},$$
$$B_n^{\{2\}}((t-x)^2;x) = B_n^{\{1\}}((t-x)^2;x) + \frac{1}{n} B_n^{\{1\}}(t-x;x) + \frac{1}{3n^2} =$$
$$= \frac{x(1+x)}{n} + \frac{1}{3n^2},$$
$$B_n^{\{2\}}((t-x)^3;x) = B_n^{\{1\}}((t-x)^3;x) + \frac{3}{2n} B_n^{\{1\}}((t-x)^2;x) +$$
$$+\frac{1}{n^2} B_n^{\{1\}}(t-x;x) + \frac{1}{4n^3} = 2\frac{x^3}{n^2} + \frac{9}{2}\frac{x^2}{n^2} + \frac{5}{2}\frac{x}{n^2} + \frac{1}{4n^3},$$
$$B_n^{\{2\}}((t-x)^4;x) = B_n^{\{1\}}((t-x)^4;x) + \frac{2}{n} B_n^{\{1\}}((t-x)^3;x) +$$
$$+\frac{2}{n^2} B_n^{\{1\}}((t-x)^2;x) + \frac{1}{5n^4} =$$
$$= \frac{3(n+2)}{n^3}x^4 + \frac{6(n+2)}{n^3}x^3 + \frac{7n+11}{n^3}x^2 + \frac{5}{n^3}x + \frac{1}{5n^4}.$$

Applying Lemma 1 we derive the following two lemmas.

Lemma 2. For every fixed $x_0 \in R_0$ *there exists a positive constant* $M_1(x_0)$ *such that for all* $n \in N$ *and* $i = 1,2$,

$$\left.\begin{array}{l} S_n^{\{i\}}((t-x_0)^4;x_0) \\ B_n^{\{i\}}((t-x_0)^4;x_0) \end{array}\right\} \le M_1(x_0)n^{-2}.$$

Lemma 3. For every fixed $x_0 \in R_0$ *and* $n \in N$ *we have*

$$nS_n^{\{i\}}(t-x_0;x_0) = nB_n^{\{i\}}(t-x_0;x_0) = \begin{cases} 0 & \text{if} \quad i=1, \\ \dfrac{1}{2} & \text{if} \quad i=2. \end{cases}$$

Moreover for $i=1,2$

$$\lim_{n\to\infty} n\, S_n^{\{i\}}((t-x_0)^2;x_0) = x_0,$$

$$\lim_{n\to\infty} n\, B_n^{\{i\}}((t-x_0)^2;x_0) = x_0(1+x_0).$$

Lemma 4 ([1]). *For every fixed* $p \in N_0$ *there exist positive constanst* $M_k(p)$, $2 \le k \le 6$ *such that for all* $x \in R_0$, $n \in N$ *and* $i=1,2$

$$\left.\begin{array}{l} w_p(x)S_n^{\{i\}}(1/w_p(t);x) \\ w_p(x)B_n^{\{i\}}(1/w_p(t);x) \end{array}\right\} \le M_2(p),$$

$$w_p(x)\,S_n^{\{1\}}((t-x)^2/w_p(t);x) \le M_3(p)\frac{x}{n},$$

$$w_p(x)\,B_n^{\{1\}}((t-x)^2/w_p(t);x) \le M_4(p)\frac{x(x+1)}{n},$$

$$w_p(x)\,S_n^{\{2\}}((t-x)^2/w_p(t);x) \le M_5(p)\frac{x+1}{n},$$

$$w_p(x)\,B_n^{\{2\}}((t-x)^2/w_p(t);x) \le M_6(p)\frac{x(x+1)+1}{n}.$$

Lemma 5([2]). *For every fixed* $q>0$ *and* $r>q$ *there exist two positive constants* $M_6(q,r)$, $M_7(q,r)$ *and a natural number* n_0,

$$n_0 > q\left(\ln\frac{r}{q}\right)^{-1}, \tag{20}$$

such that for all $x \in R_0$ *and* $n > n_0$

$$v_r(x)\,S_n^{\{i\}}(1/v_q(t);x) \le M_6(q,r), \qquad i=1,2,$$

$$v_r(x)S_n^{\{i\}}((t-x)^2/v_q(t);x) \le M_7(q,r)\begin{cases}\dfrac{x}{n} & if \quad i=1,\\ \dfrac{x+1}{n} & if \quad i=2.\end{cases}$$

2.2. Applying Lemmas 1-5 and (12) - (19) we immediately obtain the following two lemmas

Lemma 6. For fixed $p\in N_0$, $r>q>0$ *there exists positive constant* $M_8(p,q,r)$ *such that for all* $1\le m\in N$, $n_0\le n\in N$ *and* $1\le i\le 4$

$$\left\| L_{m,n}^{\{i\}}\left(\frac{1}{w_{p,q}^*(t,z)};\cdot,\cdot\right)\right\|_{p,r} \le M_8(p,q,r) \tag{21}$$

and for any $f\in C_{p,q}$

$$\left\| L_{m,n}^{\{i\}}(f;\cdot,\cdot)\right\|_{p,r} \le M_8(p,q,r)\|f\|_{p,q}. \tag{22}$$

From this and from (12) – (15) *if follows that* $L_{m,n}^{\{i\}}$, $m\ge 1$, $n\ge n_0$, $1\le i\le 4$, *is positive linear operator from the space* $C_{p,q}$ *into* $C_{p,r}$, $r>q>0$.

Proof. By (1) – (3) and (7) – (19) we can write for $(x,y)\in R_0^2$, $m,n\in N$

$$w_{p,r}^*(x,y)L_{m,n}^{\{i\}}\left(\frac{1}{w_{p,q}^*(t,z)};x,y\right)=$$

$$=\begin{cases}\left\{w_p(x)S_m^{\{i\}}\left(\dfrac{1}{w_p(t)};x\right)\right\}\left\{v_r(x)S_n^{\{i\}}\left(\dfrac{1}{v_q(t)};y\right)\right\} & \text{if} \quad i=1,2,\\ \left\{w_p(x)B_m^{\{i-2\}}\left(\dfrac{1}{w_p(t)};x\right)\right\}\left\{v_r(x)S_n^{\{i-2\}}\left(\dfrac{1}{v_q(t)};y\right)\right\} & \text{if} \quad i=3,4.\end{cases}$$

Applying now the suitable inequalities given in Lemma 4 and Lemma 5 and by (4), we immediately obtain (21) for all $m\ge 1$, $n>n_0$ and $1\le i\le 4$.

The inequality (22) follows by

$$\left\|L_{m,n}^{\{i\}}(f;\cdot,\cdot)\right\|_{p,r}\le \|f\|_{p,q}\left\|L_{m,n}^{\{i\}}\left(\frac{1}{w_{p,q}^*(t,z)};\cdot,\cdot\right)\right\|_{p,r},$$

and by (21).

3. THEOREMS

3.1. In this part we shall prove two theorems on the degree of approximation of functions $f \in C_{p,q}$ by the operators $L_{m,n}^{\{i\}}$.

Theorem 1. Suppose that $g \in C_{p,p}^{1}$ *with some* $p \in N_0$ *and* $q \in R_+$. *Then for every fixed* $r > q$ *there exist a positive constant* $M_9(p,q,r)$ *and a natural numbaer* n_0 *satisfying the condition* (20) *such that for all* $(x,y) \in R_0^2$, $m \geq 1$, $n \geq n_0$ *and* $1 \leq i \leq 4$

$$\text{(23)} \qquad w_{p,r}^{*}(x,y)\left|L_{m,n}^{\{i\}}(g;x,y) - g(x,y)\right| \leq$$

$$\leq M_9(p,q,r)\left\{\left\|\frac{\partial g}{\partial x}\right\|_{p,q}\sqrt{\frac{\varphi_i(x)}{m}} + \left\|\frac{\partial g}{\partial y}\right\|_{p,q}\sqrt{\frac{\psi_i(y)}{n}}\right\},$$

where $\varphi_1(x) = x$, $\varphi_2(x) = x+1$, $\varphi_3(x) = x(x+1)$, $\varphi_4(x) = x(x+1)+1$, $\psi_1(y) = \psi_3(y) = y$, $\psi_2(y) = \psi_4(y) = y+1$.

Proof. Let $i = 3$ and let (x,y) be a fixed point in R_0^2. For $g \in C_{p,q}^{1}$ we have

$$g(t,z) - g(x,y) = \int_x^t \frac{\partial g}{\partial u}(u,z)du + \int_y^z \frac{\partial g}{\partial v}(x,v)dv, \qquad (t,z) \in R_0^2.$$

From this and by (17) we get for $m,n \in N$

$$\text{(24)} \qquad L_{m,n}^{\{3\}}(g(t,z);x,y) - g(x,y) =$$

$$= L_{m,n}^{\{3\}}\left(\int_x^t \frac{\partial g}{\partial u}(u,z)du;x,y\right) + L_{m,n}^{\{3\}}\left(\int_y^z \frac{\partial g}{\partial v}(x,v)dv;x,y\right).$$

But by (1) – (4), we have

$$\left|\int_x^t \frac{\partial g}{\partial u}(u,z)du\right| \leq \left\|\frac{\partial g}{\partial x}\right\|_{p,q}\left|\int_x^t \frac{du}{w_{p,q}^{*}(u,z)}\right| \leq$$

$$\leq \left\|\frac{\partial g}{\partial x}\right\|_{p,q}\left(\frac{1}{w_{p,q}^{*}(t,z)} + \frac{1}{w_{p,q}^{*}(x,z)}\right)|t-x|$$

and

$$\left|\int_x^t \frac{\partial g}{\partial v}(x,v)dv\right| \le \left\|\frac{\partial g}{\partial y}\right\|_{p,q}\left(\frac{1}{w_{p,q}^*(x,z)}+\frac{1}{w_{p,q}^*(x,y)}\right)|z-y|.$$

Using the above inequalities and by (1) – (3) and (16) – (19) we get for $m,n \in N$

$$w_{p,r}^*(x,y)\left|L_{m,n}^{\{3\}}\left(\int_x^t \frac{\partial g}{\partial u}(u,z)du;x,y\right)\right| \le w_{p,r}^*(x,y)\left|L_{m,n}^{\{3\}}\left(\left|\int_x^t \frac{\partial g}{\partial u}(u,z)du\right|;x,y\right)\right| \le$$

$$\le \left\|\frac{\partial g}{\partial g}\right\|_{p,q} w_{p,r}^*(x,y)\left\{L_{m,n}^{\{3\}}\left(\frac{|t-x|}{w_{p,q}^*(t,z)};x,y\right)+L_{m,n}^{\{3\}}\left(\frac{|t-x|}{w_{p,q}^*(x,z)};x,y\right)\right\} \le$$

$$\le \left\|\frac{\partial g}{\partial x}\right\|_{p,q} v_r(y)S_n^{\{1\}}\left(\frac{1}{v_q(z)};y\right)\left\{w_p(x)B_m^{\{1\}}\left(\frac{|t-x|}{w_p(t)};x\right)+B_m^{\{1\}}(|t-x|;x)\right\}$$

and analogously

$$w_{p,r}^*(x,y)\left|L_{m,n}^{\{3\}}\left(\int_y^z \frac{\partial g}{\partial v}(x,v)dv;x,y\right)\right| \le$$

$$\le \left\|\frac{\partial g}{\partial y}\right\|_{p,q}\left\{v_r(y)S_n^{\{1\}}\left(\frac{|z-y|}{v_q(z)};y\right)+S_n^{\{1\}}(|z-y|;y)\right\}.$$

By Hölder inequality and by (16), Lemma 1, Lemma 4 and Lemma 5 we get

$$B_m^{\{1\}}(|t-x|;x) \le \left\{B_m^{\{1\}}((t-x)^2;x)\right\}^{1/2}\left\{B_m^{\{1\}}(1;x)\right\}^{1/2} \le \left\{\frac{x(1+x)}{m}\right\}^{1/2},$$

$$w_p(x)B_m^{\{1\}}\left(\frac{|t-x|}{w_p(t)};x\right) \le \left\{w_p(x)B_m^{\{1\}}\left(\frac{(t-x)^2}{w_p(t)};x\right)\right\}^{1/2} \times$$

$$\times\left\{w_p(x)B_m^{\{1\}}\left(\frac{1}{w_p(t)};x\right)\right\}^{1/2} \le M_{10}(p)\left\{\frac{x(x+1)}{m}\right\}^{1/2},$$

for all $m \in N$, and analogously for $n \ge n_0$

$$S_n^{\{1\}}(|z-y|;y) \le (y/n)^{1/2},$$

$$v_r(y)S_n^{\{1\}}\left(\frac{|z-y|}{v_q(z)};y\right)\le M_{11}(q,r)(y/n)^{1/2}.$$

Combining these, we derive from (24)

$$w_{p,r}^*(x,y)\left|L_{m,n}^{\{3\}}(g(t,z);x,y)-g(x,y)\right|\le$$

$$\le M_{12}(p,q,r)\left\{\left\|\frac{\partial g}{\partial x}\right\|_{p,q}\sqrt{\frac{x(x+1)}{m}}+\left\|\frac{\partial g}{\partial y}\right\|_{p,q}\sqrt{\frac{y}{n}}\right\}$$

for all $m\ge 1$ and $n\ge n_0$. Thus the proof of (23) for $i=3$ is completed.

The proof of (23) for $i=1,2,4$ is analogous.

Theorem 2. Assume that p, q, r, n_0 *are numbers as in Theorem 1. Then there exists a positive constant* $M_{13}(p,q,r)$ *such that for any* $f\in C_{p,q}$ *and for all* $(x,y)\in R_0^2$, $n\ge n_0$ *and* $1\le i\le 4$

$$w_{p,r}^*(x,y)\left|L_{m,n}^{\{i\}}(f;x,y)-f(x,y)\right|\le \tag{25}$$

$$\le M_{13}(p,q,r)\,\omega\left(f,C_{p,q};\sqrt{\frac{\varphi_i(x)}{m}},\sqrt{\frac{\psi_i(y)}{n}}\right)$$

where $\varphi_i(\cdot)$ *and* $\psi_i(\cdot)$ *are the functions given in Theorem 1 and* ω *is the modulus of continuity of* f *defined by* (5).

Proof. Let $f_{h,\delta}$ be the Stieklov mean of $f\in C_{p,q}$ defined by

$$f_{h,\delta}:=\frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u,y+v)\,dudv,\qquad (x,y)\in R_0^2,\quad h,\delta\in R_+.$$

We observe that

$$f_{h,\delta}(x,y)-f(x,y)=\frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,v}f(x,y)\,dudv,$$

$$\frac{\partial}{\partial x}f_{h,\delta}(x,y)=\frac{1}{h\delta}\int_0^\delta \Delta_{h,0}f(x,y+v)\,dv\equiv$$

$$\equiv\frac{1}{h\delta}\int_0^\delta\{\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y)\}\,dv,$$

$$\frac{\partial}{\partial y} f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h \Delta_{0,\delta} f(x+u,y)du \equiv$$

$$\equiv \frac{1}{h\delta}\int_0^h \{\Delta_{u,\delta} f(x,y) - \Delta_{u,0} f(x,y)\}du$$

and consequently

(26) $$\left\| f_{h,\delta} - f \right\|_{p,q} \le \omega(f, C_{p,q}; h, \delta),$$

(27) $$\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{p,q} \le 2h^{-1}\omega(f, C_{p,q}; h, \delta),$$

(28) $$\left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{p,q} \le 2\delta^{-1}\omega(f, C_{p,q}; h, \delta),$$

for all $h,\delta > 0$. This shows that if $f \in C_{p,q}$ $(p \in N_0,\ q > 0)$ then $f_{h,\delta} \in C^1_{p,q}$ for fixed $h,\delta > 0$. Hence we can write

(29) $$w^*_{p,r}(x,y)\left|L^{\{i\}}_{m,n}(f;x,y) - f(x,y)\right| \le$$

$$\le w^*_{p,r}(x,y)\Big\{\left|L^{\{i\}}_{m,n}(f(t,z) - f_{h,\delta}(t,z);x,y)\right| +$$

$$+\left|L^{\{i\}}_{m,n}(f_{h,\delta}(t,z);x,y) - f_{h,\delta}(x,y)\right| + \left|f_{h,\delta}(x,y) - f(x,y)\right|\Big\} :=$$

$$:= A_1 + A_2 + A_3$$

for $(x,y) \in R_0^2$, $m,n \in N$, $h,\delta > 0$ and $1 \le i \le 4$.

Applying (22) and (26), we get

$$A_1 \le \left\| L^{\{i\}}_{m,n}(f - f_{h,\delta};\cdot,\cdot) \right\|_{p,r} \le M_8(p,q,r)\left\| f - f_{h,\delta} \right\|_{p,q} \le$$

$$\le M_8(p,q,r)\omega(f, C_{p,q}; h, \delta)$$

for all $m \ge 1$, $n \ge n_0$, $h,\delta > 0$ and $1 \le i \le 4$.

Using Theorem 1 and (27), (28), we get

$$A_2 \le M_9(p,q,r)\left\{ \left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{p,q} \sqrt{\frac{\varphi_i(x)}{m}} + \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{p,q} \sqrt{\frac{\psi_i(y)}{y}} \right\} \le$$

$$\le M_{14}(p,q,r)\,\omega(f,C_{p,q};h,\delta)\left\{1+h^{-1}\sqrt{\frac{\varphi_i(x)}{m}}+\delta^{-1}\sqrt{\frac{\psi_i(y)}{n}}\right\}$$

$$A_3 \le \omega(f,C_{p,q};h,\delta),$$

for $(x,y)\in R_0^2$, $m\ge 1$, $n>n_0$, $h,\delta>0$ and $1\le i\le 4$, where $\varphi_i(x)$, $\psi_i(x)$ are defined in Theorem 1. Consequently we get from (29)

(30) $$w_{p,r}^*(x,y)\left|L_{m,n}^{\{i\}}(\cdot f;x,y)-f(x,y)\right|\le$$

$$\le M_{15}(p,q,r)\,\omega(f,C_{p,q};h,\delta)\left\{1+h^{-1}\sqrt{\frac{\varphi_i(x)}{m}}+\delta^{-1}\sqrt{\frac{\psi_i(y)}{n}}\right\}$$

for $(x,y)\in R_0^2$, $m\ge 1$, $n\ge n_0$, $h,\delta>0$ and $1\le i\le 4$.

Now, for fixed (x,y), m, n and i, setting $h=\sqrt{\varphi_i(x)/m}$ and $h=\sqrt{\psi_i(y)/n}$ to (30), we obtain the desired estimation (25).

From Theorem 2 we derive

Corollary 1. Let $f\in C_{p,q}$ *with some* $p\in N_0$ *and* $q\in R_+$. *Then for every* $(x,y)\in R_0^2$ *and* $1\le i\le 4$

(31) $$\lim_{n,m\to\infty} L_{m,n}^{\{i\}}(f;x,y)=f(x,y).$$

Moreover (31) *holds uniformly on every rectangle* $0\le x\le a$, $0\le y\le b$.

3.2. Now we shall prove a theorem of the Voronovskaja type

Theorem 3. Suppose that $f\in C_{p,q}^2$ *with some* $p\in N_0$ *and* $q\in R_+$. *Then for every* $(x,y)\in R_0^2$ *and* $1\le i\le 4$

(32) $$\lim_{n\to\infty} n\left\{L_{n,n}^{\{i\}}(f;x,y)-f(x,y)\right\}=\frac{x}{2}\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{y}{2}\frac{\partial^2 f}{\partial y^2}(x,y)+$$

$$+\begin{cases}0 & \text{if} \quad i=1,3,\\ \dfrac{1}{2}\dfrac{\partial f}{\partial x}(x,y)+\dfrac{1}{2}\dfrac{\partial f}{\partial y}(x,y) & \text{if} \quad i=2,4.\end{cases}$$

Proof. Let $i=1$ and let $(x_0,y_0)\in R_0^2$ be a fixed point. Then by the Taylor formula for $f\in C_{p,q}^2$ we have for $(t,z)\in R_0^2$

$$f(t,z)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(t-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(z-y_0)+$$
$$+\frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(t-x_0)^2+2\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)(t-x_0)(z-y_0)+\right.$$
$$\left.+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(z-y_0)^2\right\}+\varphi(t,z;x_0,y_0)\sqrt{(t-x_0)^4+(z-y_0)^4},$$

where $\varphi(\cdot,\cdot)\equiv\varphi(\cdot,\cdot;x_0,y_0)\in C_{p,q}$ and $\lim_{(t,z)\to(x_0,y_0)}\varphi(t,z;x_0,y_0)=0.$

From this and by (16) – (19) we derive

$$(33)\quad L_{n,n}^{\{1\}}(f(t,z);x_0,y_0)=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)S_n^{\{1\}}(t-x_0;x_0)+$$
$$+\frac{\partial f}{\partial y}(x_0,y_0)S_n^{\{1\}}(z-y_0;y_0)+\frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)S_n^{\{1\}}((t-x_0)^2;x_0)+\right.$$
$$+\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)S_n^{\{1\}}(t-x_0;x_0)S_n^{\{1\}}(z-y_0;y_0)+$$
$$\left.+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)S_n^{\{1\}}((z-y_0)^2;y_0)\right\}+$$
$$+L_{n,n}^{\{1\}}\left(\varphi(t,z)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0\right)\qquad\text{for}\quad n\in N.$$

Using the Hölder inequality, we have

$$(34)\quad \left|L_{n,n}^{\{1\}}\left(\varphi(t,z)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0\right)\right|\le$$
$$\le\left\{L_{n,n}^{\{1\}}(\varphi^2(t,z);x_0,y_0)\right\}^{1/2}\left\{L_{n,n}^{\{1\}}((t-x_0)^4+(z-y_0)^4;x_0,y_0)\right\}^{1/2}.$$

But by (16), (18) and lemma 2 we have

$$(35)\quad \left|L_{n,n}^{\{1\}}((t-x_0)^4+(z-y_0)^4;x_0,y_0)\right|=S_n^{\{1\}}((t-x_0)^4;x_0)+$$
$$+S_n^{\{1\}}((z-y_0)^4;y_0)\le M_{16}(x_0,y_0)n^{-2}\qquad\text{for}\quad n\in N.$$

Moreover the properties of function φ imply that we can apply Theorem 2 for the function $\psi(t,z)=\varphi^2(t,z)$, $\psi\in C_{2p,2q}$. Hence

$$(36)\qquad \lim_{n\to\infty} L_{n,n}^{\{1\}}(\varphi^2(t,z);x_0,y_0) = \varphi^2(x_0,y_0) = 0$$

and from (34) – (36) follows

$$(37)\qquad \lim_{n\to\infty} n\, L_{n,n}^{\{1\}}\left(\varphi(t,z)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0\right) = 0.$$

Now using (14) and Lemma 1 to (33), we immediately obtain (32) in arbitrary fixed point $(x_0,y_0)\in R_0^2$ and $i=1$.

The proof of (32) for $i=2,3,4$ is identical.

The similar results can be proved for the operators $L_{m,n}^{\{i\}}$ in the polynomial weighted spaces generated by the weight function

$$w_{p_1,p_2}(x,y)=(1+x^{p_1})^{-1}(1+y^{p_2})^{-1},\quad (x,y)\in R_0^2,\quad p_1,p_2\in N_0.$$

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