FASCICULI MATHEMATICI

Nr 30 1999

JADWIGA HACHAJ

AN APPLICATION OF THE GREEN METHOD TO A DETERMINATION OF UNIQUE SOLUTIONS OF ITERATED PARABOLIC-ELLIPTIC PROBLEMS

ABSTRACT: The aim of the paper is an application of the Green method and the Banach contraction theorem to a determination of unique classical solutions of iterated parabolic-elliptic boundary value problems.

KEY WORDS: higher-order PDE, iterated parabolic-elliptic equations, boundary value problems, the Green method, the Banach contraction theorem.

1. INTRODUCTION

In this paper we study the existence and uniqueness of classical solutions of boundary value problems for two equations

(1) $$\mathrm{PL}u(x,t) = f(x,t), \qquad (x,t) \in D,$$

and

(2) $$\mathrm{PL}u(x,t) = f(x,t,u(x,t)), \qquad (x,t) \in D,$$

where $\mathrm{P} := \mathrm{D}_x^2 - \mathrm{D}_t$, $\mathrm{L} := \mathrm{D}_x^2 - \mathrm{C}^2$, $D := (0,1) \times (0,T)$, $T > 0$ and C is a positive constant.

The problems studied in the paper have the following forms:

Problem (I): Find a unique classical solution u of equation (1) satisfying the initial condition

(3) $$\mathrm{L}u(x,0) = f_1(x), \qquad x \in (0,1),$$

and the boundary-value conditions

(4) $$u(0,t) = f_2(t), \qquad t \in (0,T],$$

(5) $$u(1,t) = f_3(t), \qquad t \in (0,T],$$

(6) $$\mathrm{L}u(0,t) = f_4(t), \qquad t \in (0,T],$$

(7) $$\mathrm{L}u(1,t) = f_5(t), \qquad t \in (0,T].$$

Problem (II): Find a unique classical solution u of equation (2) satisfying the initial condition

(8) $$\mathrm{L}u(x,0) = F_1(x), \qquad x \in (0,1),$$

and the boundary value conditions

(9) $u(0,t)=0, \quad t\in(0,T],$

(10) $u(1,t)=0, \quad t\in(0,T],$

(11) $\mathrm{L}u(0,t)=0, \quad t\in(0,T],$

(12) $\mathrm{L}u(1,t)=0, \quad t\in(0,T].$

To find the solutions of problems (I) and (II) we shall apply the suitable Green functions and the Banach contraction theorem.

The results obtained here are generalizations and continuations of some results given by Barański and Musiałek in [1], by Byszewski in [2-4], by Hachaj in [5-7] and by Krzyżański in [9].

2. THE UNIQUENESS OF A SOLUTION TO PROBLEM (I)

To prove the uniqueness of a solution to problem (I) let us consider the homogeneous equation

(13) $\mathrm{PL}u(x,t)=0, \quad (x,t)\in D,$

together with homogeneous conditions

(14) $\mathrm{L}u(x,0)=0, \quad x\in[0,1],$

(15) $u(0,t)=0, \quad t\in[0,T],$

(16) $u(1,t)=0, \quad t\in[0,T],$

(17) $\mathrm{L}u(0,t)=0, \quad t\in[0,T],$

(18) $\mathrm{L}u(1,t)=0, \quad t\in[0,T].$

Denote by $K_1(D)$ the class of all functions $u\in C^{4,2}(D)\cap C^{2,2}(\overline{D})$, satisfying equation (13) and initial-boundary conditions (14) – (18).

Let

$$D(t):=\{(x,s): x\in(0,1),\ s\in(0,t)\}, \qquad t\in(0,T],$$

and let us write equation (13) in the form

(19) $\mathrm{P}K(x,s)=0$, where $K(x,s):=\mathrm{L}u(x,s)$, $(x,s)\in D(t)$, $t\in(0,T]$.

Lemma 1. If $u\in K_1(D)$ *then* $I_1(t)+I_2(t)=0$, *for* $t\in(0,T]$, *where*

$$I_1(t):=\int_0^t\left(\int_0^1(\mathrm{D}_x^2K(x,s))K(x,s)dx\right)ds,$$

$$I_2(t) := -\int_0^t \left(\int_0^1 K(x,s)\, \mathrm{D}_s K(x,s) dx \right) ds$$

for $t \in (0,T]$.

Proof. Multiplying both sides of equation (13) by $K(x,s)$ and integrating over $D(t)$ we obtain the assertion of Lemma 1.

Lemma 2. If $u \in K_1(D)$ *then* $K = 0$ *in* $\overline{D}$.

Proof. From the integration by parts and (17) and (18), we obtain that

$$I_1(t) = \int_0^t (\mathrm{D}_x K(x,s))K(x,s)\Big|_{x=0}^{x=1} ds - \int_0^t \left(\int_0^1 (\mathrm{D}_x K(x,s))^2 dx \right) ds \le 0, \quad t \in (0,T],$$

and

$$I_2(t) = -\frac{1}{2} \int_0^1 (K(x,t))^2 dx \le 0, \qquad t \in (0,T].$$

Hence $K = 0$ in $D(t)$ for $t \in (0,T]$ and , by continuity, $K = 0$ in $\overline{D}$.

From Lemma 2, we get:

Lemma 3. If $u \in K_1(D)$ *then*

(20) $$D_x^2 u(x,s) - C^2 u(x,s) = 0 \quad \textit{for} \quad (x,s) \in D.$$

Lemma 4. If $u \in K_1(D)$ *then* $J_1(t) + J_2(t) = 0$ *for* $t \in (0,T]$, *where*

$$J_1(t) := \int_0^t \left(\int_0^1 u(x,s)\, \mathrm{D}_x^2 u(x,s) dx \right) ds,$$

$$J_2(t) := -C^2 \int_0^t \left(\int_0^1 u^2(x,s) dx \right) ds \quad \textit{for} \quad t \in (0,T].$$

Moreover, $J_1(t) \le 0$ *and* $J_2(t) \le 0$ *for* $t \in (0,T]$.

Proof. Multiplying both sides of equation (20) by $u(x,s)$ and integrating over $D(t)$, we obtain the first thesis of Lemma 4. The second thesis is a consequence of the definitions of $J_1(t)$, $J_2(t)$ for $t \in (0,T]$ and of the computations:

$$J_1(t) = \int_0^t \mathrm{D}_x u(x,s)(u(x,s))\Big|_{x=0}^{x=1} ds - \int_0^t \left(\int_0^1 (\mathrm{D}_x u(x,s))^2 dx \right) ds \le 0.$$

From Lemma 4, we have

Theorem 1. If $u \in K_1(D)$ *then* $u = 0$ *in* $\overline{D}$.

3. THE FUNDAMENTAL SOLUTION TO THE EQUATION $\mathrm{L}u = 0$

It is known (see [8], p. 245) that the function

$$U(x,y) = \frac{1}{2C} \exp(-C|x-y|)$$

is the fundamental solution to the equation $\mathrm{L}U(x,y) = 0$ in the set $D_1 :=$ $:= D_{1,1} \cup D_{1,2}$, where

$$D_{1,1} := \{(x,y): 0 < x < y < 1\}, \qquad D_{1,2} := \{(x,y): 0 < y < x < 1\}.$$

Consider the sequences

$$x_{i,0} := x \ \ (i = 1,2), \qquad x_{1,2n} := x + 2n, \ \ x_{2,2n} := x - 2n \ \ (n = 1,2,\ldots),$$

$$x_{1,2n+1} := -x - 2n, \qquad x_{2,2n+1} := -x + 2n + 2 \ \ (n = 0,1,2,\ldots).$$

Let

$$g(x,y) := U(x,y) + \sum_{n=1}^{\infty} (-1)^n (U(x_{1,n}, y) + U(x_{2,n}, y)) \tag{21}$$

and let $J := (0,1)$.

Lemma 5. The function U *satisfies the conditions:*

1. $U \in C(\overline{D}_1)$;
2. $\mathrm{D}U(x^-, x) - \mathrm{D}U(x^+, x) = 1$ *for* $x \in J$;
3. $U \in C^2(D_1)$;
4. *If* $W(x) = \int_0^1 f(y) U(x,y) dy$ *for* $x \in J$ *then, for every function* $f \in C(\overline{J})$, *the formula* $\mathrm{L}W(x) = -f(x)$, $x \in J$, *holds.*

Proof. 1. We have

$$U(x,y)=\frac{1}{2C}\exp(-C(x-y)) \quad \text{for} \quad (x,y)\in \overline{D}_{1,2},$$

$$U(x,y)=\frac{1}{2C}\exp(-C(y-x)) \quad \text{for} \quad (x,y)\in \overline{D}_{1,1}.$$

Consequently we get 1.

2. By 1, we obtain 2.

3. Evidently, $\mathrm{L}U(x,y)=0$ in D_1.

4. We have $W=W_1+W_2$, where

$$W_1(x):=\frac{1}{2C}\int_0^x f(y)\exp(-Cx+Cy)dy$$

and

$$W_2(x):=\frac{1}{2C}\int_x^1 f(y)\exp(Cx-Cy)dy.$$

By the last formulas, we obtain

$$\mathrm{D}_x W_1(x)+\mathrm{D}_x W_2(x)=-\frac{1}{2}\int_0^x f(y)\exp(-Cx+Cy)dy+$$

$$+\frac{1}{2}\int_x^1 f(y)\exp(Cx-Cy)dy,$$

$$(\mathrm{D}_x^2-\mathrm{C}^2)W(x)=-f(x)+\frac{1}{2C}\int_0^1 f(y)(\mathrm{D}_x^2-\mathrm{C}^2)\exp(-C|x-y|)=$$

$$=-f(x).$$

Indeed, let a be an arbitrary positive number. We have

$$\int_0^1 f(y)\mathrm{L}U(x,y)=\lim_{a\to 0}\left(\int_0^{x-a} f(y)\mathrm{L}U(x,y)dy+\right.$$

$$\left.+\int_{x-a}^{x+a} f(y)\mathrm{L}U(x,y)dy+\int_{x+a}^1 f(y)\mathrm{L}U(x,y)dy\right)=$$

$$=\lim_{a\to 0}\int_{x-a}^{x+a} f(y)(\mathrm{D}_x^2 U(x,y)-\mathrm{C}^2 U(x,y))dy=\lim_{x\to a}\left(\int_{x-a}^{x+a} f(y)\mathrm{D}_x^2 U(x,y)dy\right)=$$

$$= \lim_{x\to a}\left(\int_{x-a}^{x} f(y)\mathrm{D}_x^2 U(x,y)dy + \int_{x}^{x+a} f(y)\mathrm{D}_x^2 U(x,y)dy\right) =$$

$$= C^2 \lim_{a\to 0}\left(\exp(-Cx)\int_{x-a}^{x} f(y)\exp(Cy)dy + \right.$$

$$\left. + \exp(Cx)\int_{x}^{x+a} f(y)\exp(-Cy)dy\right) = 0.$$

4. Properties of the function g

By (21), we obtain the formula

$$g(x,y) := U(x,y) + H(x,y),$$

where

$$H(x,y) := \sum_{i=1}^{4} S_i(x,y),$$

$$S_1(x,y) := \sum_{n=0}^{\infty} A_n(x,y),$$

$$A_n(x,y) := \frac{1}{2C}\exp(-C(x+2n+2-y)), \qquad (x,y)\in J^2,$$

$$S_2(x,y) := \sum_{n=0}^{\infty} B_n(x,y),$$

$$B_n(x,y) := \frac{1}{2C}\exp(-C(-x+2n+2+y)), \qquad (x,y)\in J^2,$$

$$S_3(x,y) := \sum_{n=0}^{\infty} C_n(x,y),$$

$$C_n(x,y) := -\frac{1}{2C}\exp(-C(x+2n+y)), \qquad (x,y)\in J^2,$$

$$S_4(x,y) := \sum_{n=0}^{\infty} D_n(x,y),$$

$$D_n(x,y) := -\frac{1}{2C}\exp(-C(-x+2n+2-y)), \quad (x,y)\in J^2.$$

Let

$$\mathrm{D}_x^i S_j(x,y) = \sum_{n=0}^{\infty} \mathrm{D}_x^i F_n^{(j)}(x,y) \qquad (i=0,1,2;\ j=1,2,3,4), \tag{22}$$

where

$$F_n^{(1)}(x,y) := A_n(x,y), \qquad F_n^{(2)}(x,y) := B_n(x,y),$$

$$F_n^{(3)}(x,y) := C_n(x,y), \qquad F_n^{(4)}(x,y) := D_n(x,y).$$

Remark. It is evident that

$$g(x,y) = g(y,x), \qquad (x,y) \in J^2.$$

Lemma 6. If $(x,y) \in J^2$ then

$$\exp(-C\,|\,x+2n+y\,|) \le C_1 (n-1)^{-2} \quad \text{for } n>1, \tag{23}$$

where C_1 is a positive constant.

Proof. We have the inequality

$$\exp(-C\,|\,z\,|) \le C_1\,|\,z\,|^{-2}.$$

Putting in the last inequality $z = x + 2n + y$, we obtain (23).

Lemma 7. If $y \in J$ then

1. $g(0,y) = 0$;
2. $g(1,y) = 0$.

Proof. 1. It is easy to see that

$$g(0,y) = \frac{1}{2C}(\exp(-C\,|-y\,|) + \sum_{n=0}^{\infty}(\exp(-C(2n+2-y)) +$$

$$+\exp(-C(2n+2+y)) - \exp(-C(2n+y)) -$$

$$-\exp(-C(2n+2-y))) = 0.$$

2. We have

$$g(1,y) = \frac{1}{2C}(\exp(-C\,|\,1-y\,|) + \sum_{n=0}^{\infty}(\exp(-C(2n+3-y)) +$$

$$+\exp(-C(2n+1+y)) - \exp(-C(2n+1+y)) -$$

$$-\exp(-C(2n+1-y))) = 0$$

because

$$\sum_{n=0}^{\infty}(\exp(-C(2n+1-y)) = \exp(-C(1-y)) + \sum_{n=0}^{\infty}\exp(-C(2n+3-y)).$$

Lemma 8. If $(x,y)\in\overline{J^2}$ *then series* (22) *are uniformly congervent and formula* (22) *holds.*

Proof. By inequality (23), the majorants of series (22) are of the forms

$$C_{i,j}\sum_{n=2}^{\infty}(n-1)^{-2} \qquad (i=0,1,2;\ \ j=1,2,3,4)$$

and they are uniformly convergent in $\overline{J}^2$.

Consequently, we obtain the assertion of Lemma 8.

In the sequel we shall prove that

$$(24) \qquad g(x,y)\geq 0, \qquad (x,y)\in\overline{J}^2,$$

$$(25) \qquad g(x,y)\leq U(x,y), \qquad (x,y)\in\overline{J}^2.$$

To show formulas (24), (25) we shall prove some lemmas.

By [7] we obtain:

Lemma 9. The following conditions hold:

(a) *If* $x\leq y$ *then* $0\leq g(x,y)\leq g(x,x)$ *and function* g *attains its infimum* $g(0,y)=0$ *and its supremum* $g(x,y)>0$ *only for* $x=0$ *and* $x=y$.

(b) *If* $y\leq x$ *then function* g *attains its supremum for* $x=y$ *and its infimum* $g(1,y)=0$.

Lemma 10. The inequality

$$g(x,x)>0, \qquad x\in J,$$

holds.

Proof. For $x\in J$ we have

$$g(x,x)=\frac{1}{2C}\Bigg[1+\sum_{n=0}^{\infty}(\exp(-C(2n+2))+\exp(-C(2n+2))-$$

$$-\exp(-C(2x+2n))-\exp(-C(-2x+2n+2)))\Bigg]=$$

$$=\frac{1}{2C}[1-\exp(-2Cx)\sum_{n=0}^{\infty}(\exp(-C(2n))+\exp(-C(2n+2)))+$$

$$+2\sum_{n=0}^{\infty}\exp(-C(2n+2))]\geq\frac{1}{2C}(1-\exp(-2Cx))>0.$$

By the last inequality, we obtain inequalities (24) and (25).

5. Green potential

Let us consider the potential

$$V(x)=\int_0^1 f(y)g(x,y)dy=V_1(x)+V_2(x),$$

where

$$V_1(x):=\frac{1}{2C}\int_0^1 f(y)\exp(-C\,|\,x-y\,|)dy, \qquad x\in J,$$

and

$$V_2(x):=\frac{1}{2C}\int_0^1 f(y)H(x,y)dy, \qquad x\in J.$$

Lemmas 5-10 imply the following:

Lemma 11. If $f\in C(\overline{J})$ *then*

(26) $$\mathrm{L}V(x)=\mathrm{L}V_1(x)+\mathrm{L}V_2(x)=-f(x) \quad \textit{for} \quad x\in\overline{J}.$$

As a consequence of Lemmas 5-11 we get:

Theorem 2. The function g *is the Green function to equation* $\mathrm{L}u=0$ *on the interval* J *and to the homogeneous Dirichlet boundary-value conditions.*

6. Green function to the parabolic equation $Pu=0$

Let us consider the fundamental solution

$$U(x,t,y,s):=\mathrm{I}(t-s)A(t-s)^{(-1/2)}\exp\left(-\frac{(x-y)^2}{4(t-s)}\right),$$

where $A := (2\sqrt{\pi})^{-1}$ and I is the Heaviside function, to the parabolic equation

(27) $$(\mathrm{D}_x^2 - \mathrm{D}_t)U(x,t,y,s) = 0$$

in the set

$$D_2 := \{(x,t,y,s) : (x,y) \in J^2,\ 0 \le s \le t \le T\}.$$

Let us define a function G in the form

$$G(x,t,y,s) = \mathrm{I}(t-s)(U(x,t,y,s) + \sum_{n=1}^{\infty}(-1)^n (U(x_{1,n}t,y,s) + U(x_{2,n},t,y,s))$$

or in the form

$$G(x,t,y,s) = U(x,t,y,s) - H(x,t,y,s),$$

where

$$H(x,t,y,s) := -H_1(x,t,y,s) - H_2(x,t,y,s) + H_3(x,t,y,s) + H_4(x,t,y,s),$$

$$H_1(x,t,y,s) := A\sum_{n=0}^{\infty}(t-s)^{(-1/2)} \exp\left(-\frac{x+2n+2-y}{4(t-s)}\right)^2,$$

$$H_2(x,t,y,s) := A\sum_{n=0}^{\infty}(t-s)^{(-1/2)} \exp\left(-\frac{(-x+2n+2+y)^2}{4(t-s)}\right),$$

$$H_3(x,t,y,s) := A\sum_{n=0}^{\infty}(t-s)^{(-1/2)} \exp\left(-\frac{(x+2n+y)^2}{4(t-s)}\right),$$

$$H_4(x,t,y,s) := A\sum_{n=0}^{\infty}(t-s)^{(-1/2)} \exp\left(-\frac{(-x+2n+2-y)^2}{4(t-s)}\right).$$

By [9], G is the Green function to equation (27), in the domain D_2 and to the homogeneous boundary-value conditions

$$G(0,t,y,s) = G(1,t,y,s) = 0.$$

Moreover, by [9], the inequalities

$$G(x,t,y,s) \ge 0 \text{ in } D_2,$$

(28) $$G(x,t,y,s) \le U(x,t,y,s) \text{ in } D_2$$

hold.

7. REDUCTION OF PROBLEM (I) TO THOSE WITH HOMOGENEOUS BOUNDARY-VALUE CONDITIONS

Let u be a solution to problem (I). Consider a transformation given by the formula

(29) $$W(x,t) := u(x,t) - R(x,t), \quad R(x,t) := \sum_{i=1}^{4} a_i(t)x^{4-i}, \quad (x,t) \in D.$$

Define by following sets:

$$D_3 := (0,1) \times \{0\}, \qquad D_4 := \{0\} \times (0,T], \qquad D_5 := \{1\} \times (0,T].$$

It is easy to see that the lemma is true:

Lemma 12. If $f_1 \in C([0,1])$, $f_1(0) = f_1(1) = 0$, $f_i \in C([0,T])$, $(i = 2,3,4,5)$, $f_2(0) = f_4(0) = 0$, $f_3(1) = f_5(1) = 0$, *if the function* u *is a solution to problem* (I) *and if* $u \in C^{4,1}(D) \cap C^{0,0}(D)$ *then the function* W, *defined by formula* (29), *satisfies the following equation:*

(1a) $$\mathrm{PL}W(x,t) = F(x,t), \qquad (x,t) \in D,$$

where

$$F(x,t) := f(x,t) + 6C^2 a_1(t)x + 2a_2(t) + 6(D_t a_1(t))x + 2(D_t a_2(t)) -$$
$$- C^2(D_t a_1(t))x^3 - C^2(D_t a_2(t))x^2 - C^2(D_t a_3(t))x - C^2(D_t a_4(t)),$$

$(x,t) \in D$,

$$F \in \mathrm{C}^{4,1}(D) \cap \mathrm{C}^{1,0}(\overline{D} \cup D_3 \cup D_4 \cup D_5),$$

and the following conditions:

(2a) $$\mathrm{L}W(x,0) = F_1(x), \qquad x \in J,$$

where

$$F_1(x) := f_1(x) + C^2 \sum_{i=1}^{4} a_i(0)x^2 - 6a_1(0)x - 2a_2(0);$$

(3a) $$W(0,t) = u(0,t) - R(0,t) = f_2(t) - a_4(t) = 0,$$
$$f_2(t) := a_4(t), \qquad t \in (0,T];$$

(4a) $$W(1,t) = u(1,t) - R(1,t) = f_3(t) - \sum_{i=1}^{4} a_i(t) = 0,$$

$$f_3(t) := \sum_{i=1}^{4} a_i(t), \quad t \in (0,T]$$

(5a) $$\mathrm{L}W(0,t) = \mathrm{L}u(0,t) - \mathrm{L}R(0,t) = f_4(t) - 2a_2(t) + C^2 a_4(t) = 0,$$

$$f_4(t) := 2a_2(t) - C^2 a_4(t), \qquad t \in (0,T];$$

(6a) $$\mathrm{L}W(1,t) = \mathrm{L}u(1,t) - \mathrm{L}R(1,t) = f_5(t) - 6a_1(t) - 2a_2(t) + C^2 \sum_{i=1}^{4} a_i(t) =$$

$$= f_5(t) + (C^2 - 6)a_1(t) + (C^2 - 2)a_2(t) + C^2(a_3(t) + a_4(t)) = 0,$$

$$f_5(t) := (6 - C^2)a_1(t) + (2 - C^2)a_2(t) - C^2(a_3(t) + a_4(t)), \quad t \in (0,T],$$

$$a_1(t) = -(1/6)(C^2 f_2(t) - C^2 f_3(t) + f_4(t) - f_5(t)),$$

$$a_2(t) = (1/2)(C^2 f_2(t) + f_4(t)),$$

$$a_3(t) = -(1/6)(6 + 2C^2) f_2(t) - (6 - C^2) f_3(t) + 2 f_4(t) + f_5(t),$$

$$a_4(t) = f_2(t).$$

Conversely, if function W is a solution to problem (1a) - (6a) *then function* $u = W + R$ *is a solution to problem* (I).

8. Green potentials to equation (1)

Let us consider the Green potentials W_i $(i = 1,2)$, given by

(30) $$W_1(x,t) := \int_0^1 g(x,y) \left(\int_0^t \int_0^1 G(x,t,z,s) f(z,s) dz ds \right) dy,$$

(31) $$W_2(x,t) := \int_0^1 g(x,y) \left(\int_0^1 G(x,t,z,0) F_1(z) dz \right) dy.$$

Denote by $K_2(D)$ the class of all functions $F \in \mathrm{C}^{1,0}(D) \cap \mathrm{C}^{0,0}(\overline{D})$ such that

$$D_x^i F(0,t) = D_x^i F(1,t) = 0 \ \ (i = 0,1), \quad t \in (0,T]$$

and

$$F(x,0) = 0, \quad x \in (0,1).$$

Moreover, denote by $K_3(D)$ the class of all functions F^* such that

$$F^*(y,s)=F(y,s) \text{ for } (y,s)\in\overline{D} \text{ and } F^*(y,s)=0 \text{ for } (y,s)\in D_6\setminus\overline{D},$$

where

$$D_6:=\{(y,s): y\in R,\quad s\in(0,T]\}.$$

By the last definition and by (30), we obtain the formula

$$W_1(x,t):=\int_0^1 g(x,y)\int_0^t\left(\int_{-\infty}^{\infty}G(x,t,z,s)F^*(z,s)dzds\right)dy.$$

Theorem 3. If $F\in K_2(D)$ *and* $F^*\in K_3(D)$ *then:*

1. $\mathrm{PL}W_1(x,t)=F(x,t)$ for $(x,t)\in D$;
2. $\mathrm{L}W_1(x,0)=0$ for $x\in(0,1)$;
3. $W_1(0,t)=W_1(1,t)=0$ for $t\in(0,T]$;
4. $\mathrm{L}W_1(0.t)=\mathrm{L}W_1(1,t)=0,\quad t\in(0,T)$.

Proof. 1. Theorem 2 and Lemma 10 imply that:

$$\mathrm{L}W_1(x,t)=\int_0^t\int_{-\infty}^{\infty}G(x,t,y,s)F^*(y,s)dyds$$

$$\mathrm{PL}W_1(x,t)=F(x,t)=f(x,t),\quad (x,t)\in D.$$

2. We have the estimation

$$\|LW_1(x,t)\|<\sup_{\overline{D}}|F|\int_0^t\int_{-\infty}^{\infty}U(x,t,y,s)dyds$$

$$\leq C_2\int_0^t(t-s)^{(-1/2)}ds=C_3t^{(1/2)}\to 0 \quad\text{as}\quad t\to 0,$$

where C_i $(i=2,3)$ are positive constans.

3. By local uniform convergence of the integral $\mathrm{L}W_1$ in D, we have

$$W_1(0,t)=\int_0^1\lim_{x\to 0}g(x,y)\left(\int_0^t\int_{-\infty}^{\infty}G(x,t,z,s)F(z,s)dzds\right)dy=0,\qquad t\in(0,T],$$

$$W_1(1,t)=\int_0^1 \lim_{x\to 1} g(x,y)\left(\int_0^t\int_{-\infty}^{\infty} G(x,t,z,s)F(z,s)dzds\right)dy=0,\qquad t\in(0,T].$$

4. It is easy to see that

$$\mathrm{L}W_1(0,t)=\int_0^t\int_0^1 \lim_{x\to 0} G(x,t,z,s)F(z,s)dzds=0,\qquad t\in(0,T],$$

$$\mathrm{L}W_1(1,t)=\int_0^t\int_0^1 \lim_{x\to 1} G(x,t,z,s)F(z,s)dzds=0,\qquad t\in(0,T].$$

Denote by $K_4(J)$ the class of all functions F_1 such that

$$F_1\in \mathrm{C}(\overline{J}),\quad F_1(0)=F_2(1)=0\,.$$

Moreover, denote by $K_5(J)$ the class of all functions F_1^* such that

$$F_1^*=F_1 \text{ for } y\in[0,1],\qquad F_1^*=0 \text{ for } y\in R\setminus[0,1].$$

By the last formula and by (31), we obtain

$$W_2(x,t)=\int_0^t g(x,t)\left(\int_{-\infty}^{\infty} G(x,t,z,0)F_1^*(z)dz\right)dy.$$

Theorem 4. If $F_1\in K_4(J)$, $F_1^*\in K_5(J)$ *then*

1. $\mathrm{PL}W_2(x,t)=0,\quad (x,t)\in D,$
2. $\mathrm{L}W_2(x,0)=F_1(x),\quad x\in J,$
3. $W_2(0,t)=W_2(1,t)=0,\quad t\in(0,T],$
4. $\mathrm{L}W_2(0,t)=\mathrm{L}W_2(1,0)=0,\quad t\in(0,T].$

Proof. 1. We have

$$\mathrm{L}W_2(x,t)=\int_{-\infty}^{\infty} G(x,t,z,0)F_1^*(z)dz.$$

2. By [9], we obtain 2.

3. By the properties of the function G, similarly as for W_1, we get 3 and 4.

9. THE CONNECTION OF PROBLEM (1), (2) - (7) WITH PROBLEM (1A), (2A) - (7A)

By Theorems 3 and 4, we obtain

Theorem 5. If the assumptions of Theorems 3 and 4 are satisfied then the function W, *given by*

$$W(x,t) := W_1(x,t) + W_2(x,t),$$

with W_1, W_2 *defined by formulas* (30), (31), *is a solution to problem* (1a), (2a) – (7a), *and the function* u, *given by*

$$u(x,t) = W(x,t) + R(x,t),$$

is a solution to problem (1), (2) - (7).

10. A NON-LINEAR VERSION OF LINEAR PROBLEM (II)

Consider the non-linear equation

(32) $$\mathrm{PL}U(x,t) = f(x,t,U(x,t)), \quad (x,t) \in D,$$

in the domain D, satisfying the initial condition

(33) $$\mathrm{L}U(x,0) = F_1(x), \quad x \in J,$$

and the homogeneous boundary-value conditions

(34) $$U(0,t) = 0, \quad t \in (0,T],$$

(35) $$U(1,t) = 0, \quad t \in (0,T],$$

(36) $$\mathrm{L}U(0,t) = 0, \quad t \in (0,T],$$

(37) $$\mathrm{L}U(1,t) = 0, \quad t \in (0,T].$$

To solve problem (32) – (37), we shall apply the suitable integral equation and the Banach contraction theorem.

11. NOTATIONS, A DEFINITION AND LEMMAS

Let us consider the integral equation

(38) $$U(x,t) = U_0(x,t) + N(x,t,U(x,t)), \quad (x,t) \in D,$$

where

$$U_0(x,t) := W_2(x,t)$$

and

$$N(x,t,U) := \int_0^1 g(x,y)\left(\int_0^t \int_{-\infty}^{\infty} G(x,t,z,s)f(z,s,U(z,s))dzds\right)dy.$$

By K we denote a class of all functions f satisfying the conditions:

1. Function f is continuos and bounded in set $D_7 := \overline{D} \times R$;
2. Function f satisfies the Lipschitz condition

   $|f(x,t,U_1) - f(x,t,U_2)| < q|U_1 - U_2|$ for $(x,t,U_i) \in D_7$ $(i = 1,2)$,

   where $q \in (0,1)$;
3. $f(0,t,U) = f(1,t,U) = 0, \quad t \in [0,t], \quad U \in R;$
4. $f(x,t,U) = 0, \quad (x,t) \in (R \setminus J) \times [0,T], \quad U \in R;$
5. $|f(x,t,U)| < M, \quad (x,t,U) \in D_7, \quad M > 0$ is a constant.

Lemma 13. Function U_0 *satisfies the inequality*

$$|U_0(x,t)| \le \frac{1}{2C} \sup_{x \in J} |F_1(x)|, \quad t \in [0,t].$$

Lemma 14. Function N *satisfies the homogeneous boundary-value conditions*

$$N(0,t,U) = N(1,t,U) = 0 \quad \text{for} \quad t \in [0,T], \quad U \in R.$$

Proof. The above conditions follows from boundary properties of the Green function G.

12. THE BANACH SPACES OF THE CONTINUOUS FUNCTIONS $C^{0,0}(\overline{D})$

Let us consider the integral equation (38) with an unknown function U. Moreover, let us consider the Banach spaces

$$B_1 := \{U_0(x,t)\}$$

with the norm $\|U_0\| = (1/2C) \sup_{x \in J} |F_1(x)|$, and

$$B_2 = \{N(x,t,U(x,t)\}$$

with the norm, resulting from inequalities (26), (32),

$$\|N\| = \sup_{(x,t,U)\in D_7} |N(x,t,U)| = \frac{AM}{C} t^{(1/2)}.$$

13. The balls in Banach space of the continuous functions

Let

$$\mathbf{Z} := \{U_0(x,t), N(x,t,U)\} \in B_1 \times B_2,$$

where $B_1 \times B_2$ denotes the cartesian product of B_1 and B_2, be a Banach space of the continuous functions: $C^{0,0}(\overline{D})$.

By O we denote the function equal to identically zero in $\overline{D}$.

Let $K(O,r)$ denote the ball, with the centre O and radius r, of the functions u, given by

$$u(x,t) = U_0(x,t) + N(x,t,U)$$

with the norm

$$\|u\| = \|U_0 + N\| < \|U_0\| + \|N\| < r.$$

By $K(O,qr)$ we denote the ball, with the centre O and radius qr, of the functions U_0 for which

$$\|U_0\| \le qr.$$

Let $K(O,(1-q)r)$ denote the ball with the centre O and radius $(1-q)r$ of the functions N satisfying the inequality

$$\| N \| \le (1-q)r.$$

Let us define a transformation S by the formula

(39) $S : U(x,t) \to S(U_0(x,t), N(x,t,U(x,t))) := U_0(x,t) + N(x,t,U(x,t)).$

Lemma 15. The transformation S satisfies the conditions:

1. *S is approaching with the coefficient q;*
2. *S transforms the set $K(O,r)$ into itself.*

Proof. 1. Let $U_1, U_2 \in R$. We have

$$\|S(U_0(x,t), N(x,t,U_1)) - S(U_0(x,t), N(x,t,U_2))\| =$$

$$= \|N(x,t,U_1) - N(x,t,U_2)\| \le q|U_1 - U_2|.$$

2. If $U \in K(O,r)$ then

$$\|SU\| \le \|U_0\| + \|N\| \le qr + (1-q)r = r\ .$$

From the Banach contraction theorem, there exists a fixed point V to transformation S such that

$$V(x,t) = U_0(x,t) + \int_0^1 g(x,y)(\int_0^t\int_0^1 G(x,t,z,s)f(z,s,V(z,s))dzds)dy. \tag{40}$$

14. Construction of the solution to equation (40)

Let us consider the sequence $\{V_n\}_{n=1}^{\infty}$, where

$$V_0(x,t) := U_0(x,t), \quad (x,t) \in D,$$

$$V_1(x,t) := U_0(x,t) + \int_0^1 g(x,y)\left(\int_0^t\int_0^1 G(x,t,z,s)f(z,s,V_0(z,s))dzds\right)dy,$$

$$(x,t) \in D,$$

$$V_n(x,t) := U_0(x,t) + \int_0^1 g(x,y)\left(\int_0^t\int_0^1 G(x,t,z,s)f(z,s,V_{n-1}(z,s))dzds\right)dy,$$

$$(x,t) \in D\ (n = 1,2,...).$$

Since space $\boldsymbol{Z}$ is complete then

$$\lim_{n\to\infty} V_n(x,t) = V(x,t). \tag{41}$$

Consequently, function V defined by (41), satisfies equation (40).

15. Connection of the solution of the integral equation and the differential one

Theorem 6. If function V is a solution to integral equation (40) *then function V satisfies the nonlinear version of problem* (II) *i.e.:*

1. *The function* V *satisfies the equation* (32) *for* $(x,t)\in D$,
2. $\mathrm{L}V(x,0)=F_1(x), \quad x\in[0,1]$;
3. $V(0,t)=0, \quad t\in[0,T]$;
4. $V(1,t)=0, \quad t\in[0,T]$;
5. $\mathrm{L}V(0,t)=0, \quad t\in[0,T]$;
6. $\mathrm{L}V(1,t)=0, \quad t\in[0,T]$.

Proof. 1. By Theorem 3 and by [9], we obtain

$$\mathrm{L}V(x,t)=\int_0^1 G(x,t,z,0)F_1(z)dz+\int_0^t\int_0^1 G(x,t,y,s)f(y,s,V(y,s))dyds, \quad (x,t)\in D,$$

$$\mathrm{PL}V(x,t)=F(x,t,V(x,t)), \quad (x,t)\in D.$$

2. By [9] and Theorem 2, we get

$$\mathrm{L}V(x,0)=F_1(x)+\lim_{t\to 0}\int_0^t\int_0^1 G(x,t,y,s)f(y,s,V(y,s))dyds=F_1(x), \quad x\in J.$$

3–6. By the properties of functions g, G, similarly as in Theorem 3, we obtain 3–6.

REFERENCES

[1] F. Barański, J. Musiałek, The limit problems for the heat equation and for the m-dimensional cuboid, *Demonstratio Mathematica* 15(1982), 861-881.

[2] L. Byszewski, On a certain limit problem for parabolic equation in the (n+1)-dimensional time-space cube, *Comment. Math.* 25(1985), 5-20.

[3] L. Byszewski, On the homogeneous with respect to the differential equation Fourier's first linear iterated problem in the (n+1)-dimensional time-space cube, *Comment. Math.* 28(1988), 1-22.

[4] L. Byszewski, On Fourier first quasi-linear and linear iterated problems and on Fourier's second linear iterated problem in the (n+1)-dimensional time-space cube, *Comment. Math.* 28(1988), 23-45.

[5] J. Hachaj, On the Fourier problem for the rectangular parallelepiped, Cracow University of Technology, *Monograph 6*, Cracow 1978, 59-82.

[6] J. Hachaj, On the Fourier problem for the rectangular with mixed boundary data, Cracow University of Technology, *Monograph 77*, Cracow 1989, 125-134.

[7] J. Hachaj, Biparabolic Poincare problem for the quart-time plane, Cracow University of Technology, *Monograph 118,* Cracow 1991, 59-69.37
[8] E. Kamke, *Differentialgleichungen*, Vol. I, Berlin 1962.
[9] M. Krzyżański, *Partial differential equations of second order*, Vol. I, Polish Scientific Publishers, Warszawa 1971.

(Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Cracow, Poland)

Received on 25.02.1998 and, in revised form, on 04.11.1998.