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SOME REMARKS ABOUT A GENERALIZED METHOD OF SUMMABILITY OF SERIES

ABSTRACT: In this paper we investigate certain methods of summability of series. We obtain a conection between those methods.

KEY WORDS: singular differential equation, singular point, summability of series.

1. Let $G=\{g_1,\dots,g_n,\dots\}$ $(n\in N)$ be the sequence of the strictly positive functions, defined and of the class C^∞ in $[r_0,1]$, $0\le r_0<1$. By α we denote a positive constant.

Let us consider the differential operator $D_\alpha^n(G)$ defined for the functions of the class C^∞ in $[r_0,1]$ by the formulas

$$(1)\qquad \begin{cases} D^0(G)f(r)=f(r), \\ D_\alpha^i(G)f(r)=D_\alpha^{i-1}(G)f(r)+(1-r)^\alpha g_i(r)\frac{d}{dr}D_\alpha^{i-1}(G)f(r) \\ \qquad\qquad (i=1,2,\dots), \end{cases}$$

where $r\in[r_0,1]$.

In paper [7], applying the operator D_α, the following method of summability of the series introduced:

Definition. Let us consider the sequence $F=\{f_1,\dots,f_k,\dots\}$ *of the functions* f_k $(k\in N)$ *of the class* C^∞ *in* $[r_0,1]$. *The numerical series* $\sum_{k=1}^\infty a_k$, *where* a_k *are the real numbers, is summable to the sum* s *by the* $A_\alpha(n,G,F)$ *method if the series* $\sum_{k=0}^\infty f_k^{(p)}(r)a_k$, $(p=0,1,\dots,n)$ *are uniformly convergent in every interval* $[r_0,r_1]$, $r_0<r_1<1$ *and if*

$$(2)\qquad \lim_{r\to 1^-} H_\alpha(n,G,F) = s,$$

where

$$H_\alpha(n,G,F,r) = \sum_{k=0}^{\infty} D_\alpha^n(G)f_k(r)a_k.$$

It follows, from the foregoing definition and fromula (1), that

(3) $$H_\alpha(n,G,F,r) = H_\alpha(n-1,G,F,r) + (1-r)^\alpha g_n(r)\frac{d}{dr}H_\alpha(n-1,G,F,r)$$

for $r \in [r_0,1)$.

The method $A_\alpha(n,G,F)$ generalized some results of papers [2], [4], [5], [6]. In paper [7], the following theorem was shown:

Theorem 1. If a series $\sum a_n$ *is summable to sum* s $(s \in \Re)$ *by the method* $A_\alpha(n,G,F)$ *then it is summable to sum* s *by the method* $A_\alpha(n-1,G,F)$.

In the present paper we will consider the case with $s = +\infty$.

2. Let $a,b \in \Re$, $a < b$. We consider the differential equation

(4) $$(b-t)^\alpha u'(t) + [A + L(t)]u(t) = f(t), \qquad t \in [a,b),$$

where $A \in \Re$, L, f are real functions defined in $[a,b)$.

We will prove the following:

Theorem 2. Assume that:

(i) $A > 0$,

(ii) $\alpha \geq 1$,

(iii) L *is a continuous function defined in* $[a,b]$ *and* $L(b) = 0$,

(iv) f *is a continuous function defined in* $[a,b)$ *and*

$$\lim_{t \to b^-} f(t) = +\infty.$$

Then for any solution u *of equation* (4), *defined in* $[t_0,b)$, *the condition*

$$\lim_{t \to b^-} u(t) = +\infty.$$

is satisfied.

Proof. Let u be a solution of equation (4), and $[t_0,b)$ be the domain of u, where $t_0 \in [a,b)$, and $u(t_0) = u_0$.

Then function u satisfies the equation

$$u(t) = R(t,t_0)u_0 + \int_{t_0}^{t} R(t,s)(b-s)^{-\alpha} f(s)ds, \qquad t \in [t_0,b),$$

where

$$R(t,s) = \exp\left(-\int_s^t \frac{A + L(\tau)}{(b-\tau)^{\alpha}} d\tau \right)$$

for $a \le s \le t < b$.

Observe that for $\alpha \ge 1$, $\lim_{t \to b^-} R(t,t_0) u_0 = 0$.

Let M be an arbitrary positive number. From assumptiom (iv), if follows that there exists a real number $t_1 \in (t_0, b)$ such that

(5) $$f(t) \ge M, \qquad t \in (t_1, b).$$

We have

$$\int_{t_0}^{t} R(t,s)(b-s)^{-\alpha} f(s)ds = \int_{t_0}^{t_1} R(t,s)(b-s)^{-\alpha} f(s)ds +$$

$$+ \int_{t_1}^{t} R(t,s)(b-s)^{-\alpha} f(s)ds = J_1(t) + J_2(t).$$

Observe that

$$J_1 = R(t,t_0) \int_{t_0}^{t_1} R(t_0,s)(b-s)^{-\alpha} f(s)ds.$$

Since $\lim_{t \to b^-} R(t,t_0) = 0$ then $\lim_{t \to b^-} J_1(t) = 0$.

From (5), we get

$$J_2 = R(t,t_0) \int_{t_1}^{t} R(t_0,s)(b-s)^{-\alpha} f(s)ds \ge$$

$$\ge M R(t,t_0) \int_{t_1}^{t} R(t_0,s)(b-s)^{-\alpha} f(s)ds.$$

By l'Hôspital's rule and assumption (iii), we obtain

$$\lim_{t \to b^-} R(t,t_0) \int_{t_1}^{t} R(t_0,s)(b-s)^{-\alpha} f(s)ds = \lim_{t \to b^-} \frac{1}{A + L(t)} = \frac{1}{A}.$$

Hence

$$\lim_{t \to b^-} J_2(t) \ge \frac{M}{A}.$$

By the fact that M is an arbitrary positive number, we have

$$\lim_{t \to b^-} J_2(t) = +\infty.$$

Consequently

$$\lim_{t\to b^-} u(t) = +\infty.$$

Remark 1. If assumption (iv) of Theorem 2 will be replace by the assumption

$$\lim_{t\to b^-} f(t) = -\infty$$

then

$$\lim_{t\to b^-} u(t) = -\infty.$$

Remark 2. The condition $\alpha \geq 1$ in Theorem 2 is essential. For an example, let $0 < \alpha, \beta < 1$ be such that $\alpha + \beta < 1$.

Considering the equation

$$(1-t)^{-\alpha} u'(t) + (1-\alpha) u(t) = (1-t)^{-\beta},$$

we get the solution of this equation in the from

$$u(t) = C \exp((1-t)^{1-\alpha}) + \int_{t_0}^{t} (1-s)^{-(\alpha+\beta)} \exp((1-t)^{1-\alpha} - (1-s)^{1-\alpha}) ds,$$

where $C = \exp(-(1-t_0)^{1-\alpha})$.

Consequently,

$$\lim_{t\to 1^-} u(t) = C + \int_{t_0}^{t} (1-s)^{-(\alpha+\beta)} \exp(-(1-s)^{1-\alpha}) ds < +\infty.$$

However,

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 1^-} (1-t)^{-\beta} = +\infty.$$

3. We will study the problem of summability of the series, which we considered at the first part of this paper.

Assume that

$$\lim_{r\to 1^-} H_\alpha(n, G, F, r) = +\infty$$

and let

$$p(r) := H_\alpha(n, G, F, r).$$

Observe that p is the continuous function in $[r_0, 1)$.

Further, the function u, defined by the formula:

$$u(r) := H_\alpha(n-1, G, F, r), \qquad r \in [r_0, 1),$$

satisfies the equation

(6) $$(1-r)^{\alpha} g_n(r)u'(r)+u(r)=p(r).$$

Define a function h by the formula

$$h(r):=\frac{1}{g_n(r)}-\frac{1}{g_n(1)}.$$

It is easy to see that the function h is continuous in $[r_0,1]$ and $h(1)=0$.

The above formula implies that equation (6) is equivalent to the following

(7) $$(1-r)^{\alpha} u'(r)+[A+h(r)]u(r)=\frac{p(r)}{g_n(r)},$$

where $A=\frac{1}{g_n(1)}>0$.

Applying Theorem 2, we have

$$\lim_{r\to 1^-} H_{\alpha}(n-1,G,F,r)=\lim_{r\to 1^-} u(r)=\lim_{r\to 1^-}\frac{p(r)}{g_n(r)}=+\infty.$$

From the foregoing formula and the results of paper [7], we get

Theorem 3. If a series $\sum a_n$ *is summable to sum* s $(s\in\overline{\Re})$ *by the method* $A_{\alpha}(n,G,F)$ *then it is summable to sum* s *by the method* $A_{\alpha}(n-1,G,F)$.

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