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ON THE PERIODIC SOLUTION OF THE PARABOLIC PROBLEM FOR THE CYLINDER

ABSTRACT: The subject of the paper is a construction of an explicit periodic solution of the parabolic problem for the cylinder. To construct of the periodic solution we shall apply the convenient Green function. In this paper the theorems on uniqueness and existence of solution of considered (1) – (2) problem is given.

KEY WORDS: parabolic equation, periodic solution, Green function.

1. INTRODUCTION

The subject of the paper is a construction of the solution u , periodic with respect to t , of equation

$$(1) \quad Pu(x, t) = 0, \quad x = (x_1, x_2), \quad P = \Delta - D_t, \quad \Delta = D_{x_1}^2 + D_{x_2}^2$$

in the cylinder

$$D = \{(x, t) : |x| < R, \quad t \in (-\infty, \infty)\},$$

satisfying the boundary value condition

$$(2) \quad u(x, t) = g(x, t) \quad \text{for } (x, t) \in D_2 = B(D_1) \times (-\infty, \infty)$$

where

$$D_1 = \{(x, 0) : |x| < R\}, \quad B(D_1) = \{(x, 0) : |x| = R\}.$$

To the construction of the periodic solution of the problem (1) – (2) in D we shall apply the convenient Green Function.

We shall give the theorems on uniqueness and existence.

2. MOTIVATION OF CONSIDERED PROBLEM

In the monograph [3], p. 101, the similar problem for the unbounded strip is considered.

In [1] the limit problem for quasilinear diffusion equation for m – dimensional cuboid with nonlinear boundary conditions is considered.

The first Fourier problem for quasilinear parabolic equation in a cylindrical domain is considered in the paper [4].

In the papers [5] and [6] the inverse parabolic problem in a flat layer and in half-unbounded domain with curvilinear boundary is considered respectively.

3. SOME FORMULAS AND UNIQUENESS PROBLEM

By [2], pp. 510-511, the solution of the Cauchy problem with the initial condition

$$u(x, t)|_{t=T} = U(x, T)$$

and homogeneous boundary condition is of the form

$$(3) \quad u(x, t)|_{\substack{x_1=r \cos p \\ x_2=r \sin p}} = \sum_{m,n=0}^{\infty} J_n \left(l_{m,n} \frac{r}{R} \right) [A_{n,m} \cos n\psi + B_{n,m} \sin n\psi] \exp \left(\frac{l_{m,n}^2 t}{R} \right),$$

where

$$(3a) \quad A_{n,m} = \frac{a_n}{\pi r_0^2 [J'_n(l_{n,m})]^2} \int_0^R \int_0^{2\pi} U(x, T)|_{\substack{x_1=r \cos p \\ x_2=r \sin p}} J_n \left(l_{m,n} \frac{r}{R} \right) \cos(npr) dr dp$$

$$(3b) \quad B_{n,m} = \frac{2}{\pi R^2 [J'_n(l_{n,m})]^2} \int_0^R \int_0^{2\pi} U(x, t)|_{\substack{x_1=r \cos p \\ x_2=r \sin p}} J_n \left(l_{m,n} \frac{r}{R} \right) \sin(npr) dr dp,$$

$$a_n = \begin{cases} 1 & \text{for } n=0, \\ 2 & \text{for } n \neq 0, \end{cases}$$

$l_{m,n}$ are a positive solution of the equation $J_n(l) = 0$.

J_n denote the regular Bessel function of the index n .

Let

$$M(T) = \sup_{D_1} |u(x, t)|.$$

Theorem 1. If $M(T) \exp\left(-\frac{\lambda_{0,0} T}{R}\right) \rightarrow 0$ as $T \rightarrow -\infty$, then the solution u of problem (1), (2) is unique.

Proof. Let u_1, u_2 be the solutions of the problem (1), (2) and let

$$U(x, t) = u_1(x, t) - u_2(x, t), \quad M_i(T) = \sup_{D_i} |u_i(x, T)|, \quad i=1, 2.$$

We have

$$PU(x, t) = 0 \quad \text{for } (x, t) \in D, \quad U(x, t) = 0 \quad \text{for } (x, t) \in D_2$$

and

$$M(T) \leq M_1(T) + M_2(T).$$

Let us consider the solution $U(x, T)$ of the problem

$$(4) \quad PU(x, t) = 0 \quad \text{for } x \in D_1, \quad t > T,$$

with the initial condition

$$(5) \quad U(x, t)|_{t=T} = U(x, T)$$

and the homogeneous boundary condition

$$(6) \quad U(x, t) = 0 \quad \text{for } (x, t) \in B(D_1) \times (T, \infty), \quad t > T.$$

The solution of the problem (4) – (6) is of form

$$(7) \quad U(x, t) = \sum_{m, n=0}^{\infty} J_n \left(l_{m, n} \frac{r}{R} \right) [A_{m, n} \cos np + B_{n, m} \sin np] \exp \left(\frac{-l_{m, n}^2 (t - T)}{R} \right).$$

By (3a), (3b), we obtain

$$(8) \quad |A_{n, m}| \leq C_1 M(T), \quad |B_{n, m}| \leq C_2 M(T).$$

By (7), we have

$$(7a) \quad U(x, t) = S_1(x, t) + S_2(x, t),$$

where

$$S_1(x, t) = \sum_{n, m=0}^{\infty} J_n \left(l_{m, n} \frac{r}{R} \right) A_{n, m} (\cos np) \exp \left(\frac{-l_{m, n}^2 (t - T)}{R} \right),$$

$$S_2(x, t) = \sum_{\substack{n=1 \\ m=0}}^{\infty} J_n \left(l_{m, n} \frac{r}{R} \right) B_{n, m} (\sin np) \exp \left(\frac{-l_{m, n}^2 (t - T)}{R} \right).$$

By (7a), (8) we get

$$|S_1(x, t)| \leq C_1 M(T) \sum_{m, n=0}^{\infty} \exp \left(\frac{-l_{m, n}^2 (t - T)}{R} \right),$$

$$|S_2(x, t)| \leq C_2 M(T) \sum_{\substack{n=1 \\ m=0}}^{\infty} \exp \left(\frac{-l_{m, n}^2 (t - T)}{R} \right).$$

Consequently, we obtain

$$\begin{aligned}
|S_1(x, t)| &\leq C_1 M(T) \sum_{k=0}^{\infty} \exp\left(\frac{-l_k^2(t-T)}{R}\right) \leq \\
&\leq C_1 M(T) \sum_{k=0}^{\infty} \exp\left(\frac{-l_k^2(t-T)}{R}\right) \leq C_1 M(T) \sum_{k=0}^{\infty} \exp\left(\frac{-k(t-T)}{R}\right) = \\
&= C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + C_1 M(T) \sum_{k=1}^{\infty} \exp\left(\frac{-k^2(t-T)}{R}\right)
\end{aligned}$$

and

$$\begin{aligned}
|S_1(x, t)| &\leq C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + \\
&+ C_1 M(T) \left[\exp\left(\frac{-(t-T)}{R}\right) + \sum_{k=2}^{\infty} \exp\left(-\frac{k^2}{R}(t-T)\right) \right] \leq \\
&\leq C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + C_1 M(T) \left[\exp\left(\frac{-(t-T)}{R}\right) + \right. \\
&+ \sum_{k=4}^{\infty} \exp\left(-\frac{k}{R}(t-T)\right) \left. \right] \leq C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + \\
&+ C_1 M(T) \left[\exp\left(\frac{-(t-T)}{R}\right) + \frac{\exp\left(-\frac{1}{R}(t-T)\right)}{1 - \exp\left(\frac{1}{R}(t-T)\right)} \right] \leq \\
&\leq C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + C_1 M(T) \left[\exp\left(\frac{-(t-T)}{R}\right) + \right. \\
&+ C_2 \exp\left(\frac{-(t-T)}{R}\right) \left. \right] \leq C_1 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) + \\
&+ C_3 M(T) \exp\left(\frac{-(t-T)}{R}\right) \left. \right] \leq C_4 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) \rightarrow 0
\end{aligned}$$

as $T \rightarrow -\infty$.

Similarly

$$|S_2(x, t)| \leq C_4 M(T) \exp\left(\frac{-\lambda_0^2(t-T)}{R}\right) \rightarrow 0 \quad \text{as } T \rightarrow -\infty.$$

Thus $U(x, t) = 0$ and $u_1(x, t) = u_2(x, t)$ for $(x, t) \in D$.

4. GREEN FUNTION

By [8] there exists the Green function

$$G(x, t; y, s) = \Gamma(x, t; y, s) - H(x, t; y, s).$$

where

$$\Gamma(x, t; y, s) = \frac{1}{t-s} \exp(B(t, s)r^2(x, y)),$$

$$r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2, \quad B(y, s) = \frac{-1}{4(t-s)},$$

$$H(x, t, y, s) = \int_{-\infty}^t \int_{B(D_1)} \Gamma(x, t; \bar{y}, \bar{s}) \Phi(\bar{y}, \bar{s}; y, s) dS_{\bar{y}} d\bar{s}$$

and the function Φ satisfies the integral equation

$$D_n H(x, t; y, s) \rightarrow A \Phi(x, t; y, s) + \int_s^t \int_{B(D_1)} D_n \Gamma(x, t; \bar{y}, \bar{s}) \Phi(\bar{y}, \bar{s}; y, s) dS_{\bar{y}} d\bar{s}$$

5. THE SOLUTION OF THE PROBLEM (1), (2)

Let us consider the Green potential

$$u(x, t) = A \int_{-\infty}^t \int_{B(D_1)} g(y, s) D_{n(x)} G(x, t; y, s) dS_y ds.$$

Lemma 1. If $g(y, s)$ is continuous and bounded in the set D_2 , then:

- 1° the function u satisfies the equation (1) for $(x, t) \in D$,
- 2° the function u satisfies the boundary condition:

$$\lim_{x \rightarrow X \in B(D_1)} u(x, t) = g(X, t) \quad \text{for } (X, t) \in D_2.$$

Proof. Ad 1°. By [7], p. 137, we obtain the condition 1° of our Lemma.
Ad 2°. We have

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

here

$$u_1(x, t) = \int_{-\infty}^t \int_{B(D_1)} g(y, s) D_n \left[\int_{-\infty}^t \int_{B(D_1)} \Gamma(x, t; \bar{y}, \bar{s}) \Phi(\bar{y}, \bar{s}; y, s) dS_{\bar{y}} d\bar{s} \right] dy ds$$

$$\xrightarrow{x \rightarrow X \in B(D_1)} g(X, t) + A \int_{-\infty}^t \int_{B(D_1)} g(y, s) D_{n(x)} \Gamma(X, t, y, s) dS_y ds$$

and

$$u_2(x, t) = A \int_{-\infty}^t \int_{B(D_1)} g(y, s) D_n \left[\int_{-\infty}^t \int_{B(D_1)} \Gamma(x, t; \bar{y}, \bar{s}) \Phi(\bar{y}, \bar{s}; y, s) dS_{\bar{y}} d\bar{s} \right] dy ds$$

$$\xrightarrow{x \rightarrow X \in B(D)} A \int_{-\infty}^t \int_{B(D_1)} g(y, s) \left[\Phi(X, t) + \int_{-\infty}^t \int_{B(D_1)} D_n \Gamma(x, t; \bar{y}, \bar{s}) \times \right.$$

$$\left. \times \Phi(\bar{y}, \bar{s}; y, s) dS_{\bar{y}} d\bar{s} \right] dy ds = A \int_{-\infty}^t \int_{B(D_1)} g(y, s) D_{n(x)} \Gamma(X, t, y, s) dy ds.$$

Thus

$$u_1(X, t) - u_2(X, t) = g(X, t) + A \int_{-\infty}^t \int_{B(D_1)} g(y, s) [D_{n(x)} \Gamma(X, t; y, s) -$$

$$- D_{n(x)} \Gamma(X, t; y, s)] dS_y ds = g(X, t) + 0 = g(X, t).$$

Hence we obtain

$$U(X, t) = g(X, t) \quad \text{for } X \in B(D_1), \quad t \in (-\infty, \infty).$$

6. EXISTENCE AND UNIQUENESS THEOREM

By Theorem 1 and by Lemma 1, we obtain

Theorem 2. If $g(y, s)$ is continuous and bounded in D_2 , then the function u defined by formula (3) is the unique solution of the problem (1) – (2).

7. FUNDAMENTAL THEOREM CONCERNING THE PERIODIC SOLUTION

By Theorem 2 we obtain

Theorem 3. If g is continuous and bounded in D_2 and g is periodic with respect to t , i.e. $g(x, t+p) = g(x, t)$, then:

- 1° the solution u of the problem (1), (2) is periodic,
- 2° the function $D_t u(x, t)$ is periodic with respect to t ,
- 3° the function $\Delta u(x, t)$ is periodic with respect to t .

Proof. Ad 1°. By Theorems 1 and 2 there exists the unique solution u of the problem (1), (2), i.e.

$$Pu(x, t) = 0 \quad \text{for } (x, t) \in D, \quad u(x, t) = g(x, t) \quad \text{for } (x, t) \in D_2.$$

Let $U(x, t) = u(x, t+p)$. The function U satisfies the equation $PU(x, t) = Pu(x, t+p) = 0$ for $(x, t) \in D$ and the boundary conditions $U(x, t) = g(x, t+p)$.

Since $g(x, t+p) = g(x, t)$ for $(x, t) \in D_2$, thus $U(x, t) = g(x, t)$ for $(x, t) \in D_2$, $PU(x, t) = 0$ for $(x, t) \in D$ and $U(x, t) = g(x, t)$ for $(x, t) \in D_2$.

By Theorem 1, we obtain

$$U(x, t) = u(x, t+p) = u(x, t) \quad \text{for } (x, t) \in D.$$

Ad 2°. By 1° we have

$$\begin{aligned} D_t u(x, t) &= \lim_{h \rightarrow 0} \frac{u(x, t+h) - u(x, t)}{h} = \lim_{h \rightarrow 0} \frac{u(x, t+p+h) - u(x, t+p)}{h} = \\ &= D_t u(x, t+p) \quad \text{for } (x, t) \in D. \end{aligned}$$

Ad 3°. By 1°, 2° and by (1) we obtain

$$\Delta u(x, t+p) = D_t u(x, t+p) = D_t u(x, t) = \Delta u(x, t) \quad \text{for } (x, t) \in D.$$

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