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THE PERIODIC SOLUTION TO BIPARABOLIC EQUATION FOR THE THREE – DIMENSIONAL TEMPORAL – SPATIAL CYLINDER WITH BOUNDARY CONDITIONS OF RIQUIER TYPE

ABSTRACT: In this paper we shall construct the periodic solution for biparabolic equation in cylindrical domain with boundary conditions of Riguier type.

To construct the solution we shall apply the convenient heat biparabolic potentials with unkow densities and biparabolic potential of the source function. To determine these densities we shall apply the suitable system of integral equations.

The periodic solution is a sum of periodic boundary potentials and periodic biparabolic potential of the source function.

KEY WORDS: biparabolic equation, periodic solution, biparabolic potentials.

1. INTRODUCTION

The subject of the paper is the construction to the periodic solution u of the equation

(1) $$P^2u(x,t) = f(x,t), \quad x = (x_1, x_2), \qquad P^2 = P(P),$$
$$P = \Delta - D_t, \qquad G = D_{x_1}^2 + D_{x_2}^2$$

in the domain

$$D = \{(x,t) : x \in D_1,\, t \in (-\infty, \infty)\}, \qquad D_1 = \{(x,0) : |x| < R\},$$

satisfying the boundary conditions

(2) $$P^i u(x,t) = h_i(x,t), \quad (x,t) \in S = \{(x,t) : (x,t) \in B(D_1) \times (-\infty,\infty)\},$$
$$B(D_1) = \{(x,0) : |x| = R\}, \quad i = 0,1,$$

and the periodicity condition

(3) $$u(x, t+p) = u(x,t), \qquad (x,t) \in D,$$

where $p > 0$ is the period.

The functions f, h_i, $i = 0,1$, are given.

2. MOTIVATION OF THE CONSIDERED PROBLEM

In [5] and [6] the bicaloric limit problems in cylindrical domain and for the strip with conditions compatible with differential operator are considered.

In [2] the periodic solution for parabolic limit problem in cylindrical domain is given.

In [3] the periodic solution for biparabolic limit problem for the strip with boundary conditions of Riquier type is given.

The parabolic limit problem in temporal-spatial strip with boundary conditions of Lauricella type is considered in [1].

3. SOME DEFINITION

Definition 1. Denote by (K_1) *the class of all functions* $f: D \to \Re$, *such that* $f \in C^{1,0}(D)$ *and comp.* $\operatorname{supp} f \subset D$.

Definition 2. Denote by (K_2) *the class of all functions* $h: S \cup S_1 \to \Re$, *such that* $h \in C^{2,1}(S \cup S_1)$,

$$S_1 = \{(x,t) : 0 < R_1 < |x| \le R,\ t \in (-\infty, \infty)\}.$$

Definition 3. Denote by (K_3) *the class of all functions* $u: D \to \Re$, *such that* $u \in C^{4,2}(D) \cap B(D)$, *where* $B(D)$ *denote the class of all bounded functions in the domain* D.

4. THE BIPARABOLIC POTENTIALS COMPATIBLE WITH BOUNDARY CONDITIONS FOR THE TEMPORAL-SPATIAL CYLINDER

Let

$$U(x,t;y,s) = (t-s)^{-1} \exp(B(t,s) r^2(x,y));$$

$$B(t,s) = (-4(t-s))^{-1}, \qquad r^2(x,y) = \sum_{i=1}^{2} (x_i - y_i)^2.$$

Let us consider the following potentials

$$w_k(x,t) = A_k \int_{-\infty}^{t} \int_{B(D_1)} q_k(y,s)(t-s)^k D_{n(y)} U(x,t;y,s)\, dS(y)\, ds,$$

where

$$A_k = ((k+1)!2\sqrt{\pi})^{-k+1}, \qquad k = 0,1.$$

5. PROPERTIES OF THE POTENTIALS w_k, $k = 0,1$

Theorem 1. If $h_i \in (K_2)$ *and is periodic, i.e.* $h_i(x,t+p) = h_i(x,t)$, $i = 0,1$, *then*

1° $P^2 w_k(x,t) = 0$ for $(x,t) \in D$,

$$2° \; P^i w_k(x,t) = \begin{cases} A_k \int\limits_{-\infty}^{t} \int\limits_{B(D_1)} q_k(y,s)(t-s)^{k-1} D_{n(y)} U(X,t;y,s) dS(y) ds & for \; i < k, \\ q_k(X,t) + \int\limits_{-\infty}^{t} \int\limits_{B(D_1)} q_k(y,s) D_{n(y)} U(X,t;y,s) dS(y) ds & for \; i = k, \\ 0 \quad as \quad (x,t) \to (X,t) & for \; i > k, \end{cases}$$

where $X \in S$ and symbol $D_{n(y)}$ denote normal inward, i, $k \in \{0,1\}$,

3° $w_k(x,t+p) = w_k(x,t)$, $(x,t) \in D$, if q_k are periodic. (Periodicity densities q_k we shall prove below.)

Proof. The proofs of the assertions 1°, 2° are similar to the proof of Lemma 2 in [5].

Ad 3°. To verify the periodicity of the function $w(x,t) = \sum_{k=0}^{1} w_k(x,t)$ we shall prove that the functions $w_k(x,t)$, $k = 0,1$, are periodic.

For the integral w_k we have

$$w_k(x,t+p) = A_k \int\limits_{-\infty}^{t+p} \int\limits_{B(D_1)} q_k(y,s)(t+p-s) D_{n(y)} U(x-y; t+p-s) dS(y) ds.$$

Applying to change of the integral variable

$$s = z + p, \quad ds = dz, \quad \text{if } s \in (-\infty, t+p), \quad \text{then } z \in (-\infty, t),$$

we obtain

$$w_k(x,t+p) = A_k \int\limits_{-\infty}^{t} \int\limits_{B(D_1)} q_k(y,z+p)(t+p-p-z) \times$$

$$\times D_{n(y)} U(x-y; t+p-p-z) dS(y) dz,$$

and if $q_k(y,z+p) = q_k(y,z)$, then we obtain that

$w_k(x,t+p) = w_k(x,t)$, $(x,t) \in D$ (Periodicity q_k we shall prove below.)

6. The potential compatible with source function and its properties

Let us consider the following potential

$$w_2(x,t) = A_2 \int_{-\infty}^{t} \iint_{D_1} f(y,s)(t-s)U(x,t;y,s)dyds,$$

$$A_2 = (6\sqrt{\pi})^{-1}.$$

Let

$$w_3^i(x,t) = A_3 \int_{-\infty}^{t} \iint_{D_1} f(y,s)(t-s)^{1-i}U(X,t;y,s)dyds, \quad i = 0,1.$$

Theorem 2. If $f \in (K_1)$ *and f is periodic, i.e.* $f(x,t+p) = f(x,t)$, *then*

1° $P^2 w_2(x,t) = f(x,t), \quad (x,t) \in D,$

2° $P^i w_2(x,t) \to A_2 \int_{-\infty}^{t} \iint_{D_1} f(y,s)(t-s)U(X,t;y,s)dyds$

as $(x,t) \to (X,t)$ *and* $P^i w_2(X,t) \in C(S), \quad i = 0,1,2,$

3° $w_2(x,t+p) = w_2(x,t), \quad (x,t) \in D.$

Proof. The proofs of the assertions 1° and 2° are similar to the proof of Lemma 3 in [1].

Ad 3°. For the integral w_2, we have

$$w_2(x,t+p) = A_3 \int_{-\infty}^{t+p} \iint_{D_1} f(y,s)(t+p-s)U(x,y,t+p-s)dyds.$$

Applying the change of the itegral variable

$$s = z + p, \quad ds = dz, \quad \text{if } s \in (-\infty, t+p), \quad \text{then } z \in (-\infty, t),$$

we obtain

$$w_2(x,t+p) = A_2 \int_{-\infty}^{t} \iint_{D_1} f(y,z+p)(t+p-z-p)U(x-y;t+p-z-p)dyds,$$

and by periodicity *f*, we obtain

$$w_3(x,t+p) = w_3(x,t), \qquad (x,t) \in D.$$

7. THE INTEGRAL EQUATION COMPATIBLE WITH BOUNDARY CONDITIONS

Suppose, that the solution of problem (1) – (3) is of from

$$w(x,t) = \sum_{i=0}^{2} w_i(x,t), \tag{4}$$

where the densities q_i, $i = 0,1$, are unknown functions.

By (2) we obtain the system of integral equations for unknown function q_i, $i = 0,1$, being of the form

$$P^i w(X,t) = K_i(X,t), \qquad i = 0,1,$$

or

$$(5_0) \quad q_1(X,T) + \int_{-\infty}^{t} \int_{B(D_1)} q_1(y,s) C_1^1 D_{n(y)} U(X,t;y,s) dS(y) ds = \overline{K}_1(X,t),$$

$$(5_1) \quad q_0(X,T) + \int_{-\infty}^{t} \int_{B(D_1)} q_0(y,s) C_0^1 D_{n(y)} U(X,t;y,s) dS(y) ds +$$

$$+ \int_{-\infty}^{t} \int_{B(D_1)} q_1(y,s) C_1^2 (t-s) D_{n(y)} U(X,t;y,s) ds = \overline{K}_0(X,t),$$

where

$$\overline{K}_i(X,t) = K_i(X,t) - w_2^i(X,t), \qquad i = 0,1,$$

and C_i^k denote a suitable positive constans.

8. SOLUTION OF THE SYSTEM OF THE INTEGRAL EQUATION COMPATIBLE WITH BOUNDARY CONDITIONS

By (5_0) we determine q_1. Substituting q_1 in (5_1) we determine q_0. Consequently we obtain the equivalent system to $(5_0) - (5_1)$ being of the form

$$(6_0) \quad q_1(X,T) = \overline{K}_1(X,t) + \int_{\infty}^{t} \int_{B(D_1)} q_1(y,s)(-C_1^1) D_{n(y)} U(X,t;y,s) dS(y) ds,$$

$$(6_1)\quad q_0(X,T)=\overline{K}_0(X,t)+\int_{-\infty}^{t}\int_{B(D_1)} q_0(y,s)C_0 N(X,t;y,s)dS(y)ds$$

where $\overline{K}_j(X,t)$, $j=0,1,2$, are functions of the class $C(\overline{S})$ and

$$N(X,t;y,s)=D_{n(y)}U(X,t;y,s).$$

By [4], p. 132, the kernel N has the from

$$N(X,t;y,s)=C[r(X,y)]^{2,85}(t-s)^{-1,425}[r(X,y)]^{-0,85}(t-s)^{-0,575.}$$

Let

$$N_0(X,t;y,s)=N(X,t;y,s),$$

$$N_1(X,t;y,s)=\int_{-\infty}^{t}\int_{B(D_1)} N(X,t;z,r)N(z,r,y,s)dS(z)dr,$$

$$N_{i+1}(X,t;y,s)=\int_{-\infty}^{t}\int_{B(D_1)} N(X,t;z,r)N_i(z,r,y,s)dS(z)dr,\quad i=1,2,3,...$$

Let

$$R(X,t;y,s,\lambda)=\sum_{n=0}^{\infty}\lambda^n N_n(X,t;y,s).$$

Let us consider system of the equations

$$(6_{a_i})\quad q_{2-i}(X,t)=\overline{K}(X,T)+\lambda\int_{-\infty}^{t}\int_{B(D_1)} q_{2-i}(y,s)N(X,t;y,s)dS(y)ds,\quad i=1,2.$$

By [4], vol. II, p. 132, we obtain the following result:

Theorem 3. If $h_{2-i}\in(K_2)$, $i=1,2,3$, $f\in(K_1)$, *then a unique solution of the system* (6_{a_i}) *are the functions*

$$q_{2-i}(X,t,\lambda)=\overline{K}_{2-i}(X,T)+\int_{-\infty}^{t}\int_{B(D_1)}\overline{K}_{2-i}(y,s)R(X,t;s,\lambda)dS(y)ds,\quad i=1,2,$$

and

$$q_{2-i}(X,t,1)=q_{2-i}(X,t),\quad i=1,2,$$

is the unique solution of equation (6_i) $i=1,2$, *belonging to the class* $C(\overline{S})$.

9. PERIODICITY OF THE DENSITIES q_i, $i = 0,1,2$

Theorem 4. The densities q_i, $i = 0,1$, *are periodic with respect to t, i.e.* $q_i(t) = q_i(t+p)$, $i = 0,1$.

Proof. By periodicity of the functions $h_i(x,t)$, and by point 3° of the Theorem 2 of course the functions $\overline{K_i}$ are periodic with respect to variable t. To verify of periodicity the densities q_{2-i}, $i = 1,2$ we shall prove that following equation

$$\int_{-\infty}^{t}\int_{B(D_1)} R(X,t;y,s)dS(y)ds = \int_{-\infty}^{t}\int_{B(D_1)} R(X,t+p;y,s)dS(y)ds$$

holds i.e.

$$\int_{-\infty}^{t}\int_{B(D_1)}\sum_{n=0}^{\infty} N_n(X,t;y,s)dS(y)ds = \int_{-\infty}^{t}\int_{B(D_1)}\sum_{n=0}^{\infty} N_n(X,t+p;y,s)dS(y)ds.$$

For $n = 0$ we have

$$\int_{-\infty}^{t}\int_{B(D_1)} N_0(X,t+p;y,s)dS(y)ds = \int_{-\infty}^{t}\int_{B(D_1)} D_{n(y)}U(X-y,t+p-s)dS(y)ds.$$

Applying the change of the integral variable

$$s = p+z,\quad ds = dz,\quad \text{if } s \in (-\infty, t+p),\quad \text{then } z \in (-\infty, t),$$

we obtain

$$\int_{-\infty}^{t}\int_{B(D_1)} N_0(X,t+p;y,s)dS(y)ds = \int_{-\infty}^{t}\int_{B(D_1)} D_{n(y)}U(X-y;t-z)dS(y)dz =$$

$$= \int_{-\infty}^{t}\int_{B(D_1)} D_{n(y)}U(X-y;t-s)dS(y)ds.$$

For $n = 1$, we have

$$\int_{-\infty}^{t}\int_{B(D_1)}\left[N_0(x,t;y,z)dS(y)ds + \int_{-\infty}^{t}\int_{B(D_1)} N(X,t;z,s)N(z,s;y,s_1)dS(z)ds\right].$$

Since

$$\int_{-\infty}^{t}\int_{B(D_1)} N_0(X,t;y,z)dS(y)ds \quad \text{is periodicity thus to be sufficent}$$

to exhibit that

$$\int_{-\infty}^{t}\int_{B(D_1)} N(x,t;z,s)N(z,s;y,s)dS(z)ds \quad \text{is periodic.}$$

Let us consider the integral

$$\int_{-\infty}^{t+p}\int_{B(D_1)} N(x,t+p;z,s)N(z,s;y,s_1)dS(z)ds =$$

$$= \int_{-\infty}^{t+p}\int_{B(D_1)} D_{n(y)}U(x-z;t+p-s)D_{n(y)}U(z-y,s-s_1)dS(z)ds.$$

Applying in the last integral the change of the integral variable

$$s = p+z, \quad ds = dz, \quad \text{if} \quad s\in(-\infty,t+p), \quad \text{then} \quad z\in(-\infty,t),$$

we obtain

$$\int_{-\infty}^{t}\int_{B(D_1)} D_{n(y)}U(x-z;t+p-p-z)D_{n(y)}U(z-y,p+z-s_1)dS(z)dz.$$

Applying in the last integral the change of the integral variable

$$s_1 = p+z_1, \quad ds_1 = dz_1, \quad \text{if} \quad s_1\in(-\infty,p+z), \quad \text{then} \quad z_1\in(-\infty,z),$$

we get

$$\int_{-\infty}^{z}\int_{B(D_1)} D_{n(y)}U(x-z;t-z)D_{n(y)}U(z-y+z-z_1)dz_1.$$

Consequently we obtain periodicity of the integral

$$\int_{-\infty}^{t}\int_{B(D_1)} N(x,t;z,s)N(z,s;y,s_1)dS(z)ds_1.$$

Similarly for $n>1$ applying to the integral

$$\int_{-\infty}^{t}\int_{B(D_1)} R(X,t+p;y,s)dS(y)ds$$

the sequence of the changes of the integral variables

$$s_j = p + z_j,\ ds_j = dz_j, \text{ if } s_j \in (-\infty, p + z_j), \text{ then } z_j \in (-\infty, z_{j-1})\ \ j = 1,2,3,4...,$$

we get

$$\int_{-\infty}^{t} \int_{B(D_1)} R(X, t + p; y, s) dS(y) ds = \int_{-\infty}^{t} \int_{B(D_1)} R(X, t; y, s) dS(y) ds.$$

Consequently we obtain periodicity of the functions q_i, $i = 0,1$.

10. Solution of the problem (1) – (3)

By Theorem 1 – 4, we obtain the following result

Theorem 5. If the assumptions of Theorems 1 – 4 are satisfied, then the function

$$u(x,t) = \sum_{i=0}^{2} w_i(x,t)$$

is a periodic solution of problem (1) – (3).

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