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INTEGRAL INEQUALITIES OF WIRTINGER AND OPIAL TYPE IN THREE INDEPENDENT VARIABLES

ABSTRACT: In the present paper we establish some new Wirtinger and Opial type integral inequalities involving functions of three independent variables and their partial derivatives. The method used in the proof is elementary and our results provide new estimates on inequalities of this type.

KEY WORDS: integral inequalities, functions of three independent variable, Hölder's inequality, Schwarz inequality, Fubini's theorem.

1. INTRODUCTION

Integral inequalities of the Wirtinger and Opial type involving functions of one independent variable and their derivatives have been established by many authors, see [2, 3, 4] and the references given therein. Many independent variable integral inequalities of Wirtinger and Opial type have an interest in their own right and also have significant applications in mathematical analysis. Some recent contributions in this direction containing a number of new inequalities of Wirtinger and Opial type can be found in [1, 5, 7-10]. The main purpose of the present paper is to establish some new integral inequalities of the Wirtinger and Opial type involving functions of three independent variables and their partial derivatives. An interesting feature of the inequalities established in this paper is that the analysis used in their proofs is quite elementary and our results provide new estimates on this type of inequalities.

2. STATEMENT OF RESULTS

In what follows R denotes the set of real numbers. We use the notation $\Delta = [a,k]\times[b,m]\times[c,n]$ for a, b, c, k, m, n in R. If $f(r,s,t)$ is a differentiable function defined on Δ, then its partial derivatives are denoted by $D_1 f(r,s,t) = \frac{\partial}{\partial r} f(r,s,t)$, $D_2 f(r,s,t) = \frac{\partial}{\partial s} f(r,s,t)$, $D_3 f(r,s,t) = \frac{\partial}{\partial t} f(r,s,t)$ and $D_3 D_2 D_1 f(r,s,t) = \frac{\partial^3}{\partial t\,\partial s\,\partial r} f(r,s,t)$. We denote by $F(\Delta)$ the class of continuous functions $f:\Delta \to R$ for which $D_1 f(r,s,t)$, $D_2 f(r,s,t)$, $D_3 f(r,s,t)$, $D_3 D_2 D_1 f(r,s,t)$ exist and continuous on Δ and such that $f(a,s,t) = f(k,s,t) =$

$= f(r,b,t) = f(r,m,t) = f(r,s,c) = f(r,s,n) = 0$ for $a \le r \le k$, $b \le s \le m$, $c \le$ $\le t \le n$.

Our main results given in this paper are established in the following theorems.

Theorem 1. Suppose that $f, g \in F(\Delta)$. *Then*

$$(2.1) \quad \int_a^k \int_b^m \int_c^n |f(r,s,t)||g(r,s,t)| dt ds dr \le \frac{1}{2}\left[\frac{(k-a)(m-b)(n-c)}{8}\right]^2 \times$$
$$\times \int_a^k \int_b^m \int_c^n \left[|D_3 D_2 D_1 f(r,s,t)|^2 + |D_3 D_2 D_1 g(r,s,t)|^2\right] dt ds dr,$$

$$(2.2) \quad \int_a^k \int_b^m \int_c^n \big[|f(r,s,t)||D_3 D_2 D_1 g(r,s,t)| +$$
$$+ |g(r,s,t)||D_3 D_2 D_1 f(r,s,t)|\big] dt ds dr \le$$
$$\le \frac{(k-a)(m-b)(n-c)}{8} \int_a^k \int_b^m \int_c^n \big[|D_3 D_2 D_1 f(r,s,t)|^2 +$$
$$+ |D_3 D_2 D_1 g(r,s,t)|^2 \big] dt ds dr.$$

Remark 1. We note that, in the special case when $g(r,s,t) = f(r,s,t)$ for $(r,s,t) \in \Delta$, the inequalities (2.1) and (2.2) reduces respectively to the following Wirtinger type integral inequality

$$(2.3) \quad \int_a^k \int_b^m \int_c^n |f(r,s,t)|^2 dt ds dr \le \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^2 \times$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(r,s,t)|^2 dt ds dr,$$

and Opial type integral inequality

$$(2.4) \quad \int_a^k \int_b^m \int_c^n \big[|f(r,s,t)||D_3 D_2 D_1 f(r,s,t)| dt ds dr \le$$
$$\le \frac{(k-a)(m-b)(n-c)}{8} \int_a^k \int_b^m \int_c^n \big[|D_3 D_2 D_1 f(r,s,t)|^2 dt ds dr,$$

involving functions of three independent variables.

Theorem 2. Let $1 < p_i < \infty$, *for* $i = 1,2,3$ *be constants and suppose that* $f, g, h \in F(\Delta)$. *Then*

$$(2.5)\quad \int_a^k\int_b^m\int_c^n \Big[|f(r,s,t)|^{p_1}|g(r,s,t)|^{p_2} + |g(r,s,t)|^{p_2}|h(r,s,t)|^{p_3} + |h(r,s,t)|^{p_3}|f(r,s,t)|^{p_1}\Big]dt\,ds\,dr \le$$

$$\le \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{2p_1}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 f(r,s,t)|^{2p_1}\,dt\,ds\,dr + \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{2p_2}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 g(r,s,t)|^{2p_2}\,dt\,ds\,dr + \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{2p_3}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 h(r,s,t)|^{2p_3}\,dt\,ds\,dr,$$

$$(2.6)\quad \int_a^k\int_b^m\int_c^n |f(r,s,t)|^{p_1}|g(r,s,t)|^{p_2}|h(r,s,t)|^{p_3} \times \Big[|f(r,s,t)|^{p_1} + |g(r,s,t)|^{p_2} + |h(r,s,t)|^{p_3}\Big]dt\,ds\,dr \le$$

$$\le \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{4p_1}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 f(r,s,t)|^{4p_1}\,dt\,ds\,dr + \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{4p_2}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 g(r,s,t)|^{4p_2}\,dt\,ds\,dr + \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^{4p_3}\int_a^k\int_b^m\int_c^n |D_3D_2D_1 h(r,s,t)|^{4p_3}\,dt\,ds\,dr.$$

Remark 2. If we take $p_1 = p_2 = p_3 = 1$ and $g(r,s,t) = h(r,s,t) = f(r,s,t)$ in (2.5) and (2.6), then we get respectively the inequality (2.3) and the following Wirtinger type integral inequality

$$(2.7)\quad \int_a^k\int_b^m\int_c^n |f(r,s,t)|^4\,dt\,ds\,dr \le$$

$$\le \left[\frac{(k-a)(m-b)(n-c)}{8}\right]^4 \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(r,s,t)|^4 \, dt\, ds\, dr.$$

For a detailed discussion regarding the Wirtinger type inequalities see [1-5].

Theorem 3. Let $1 < p_i < \infty$, *for* $i = 0,1,2,3$ *be constants and suppose that* $f \in F(\Delta)$. *Then*

$$\text{(2.8)} \quad \int_a^k \int_b^m \int_c^n |f(r,s,t)|^{p_0} |D_1 f(r,s,t)|^{p_1} |D_2 f(r,s,t)|^{p_2} |D_3 f(r,s,t)|^{p_3} \, dt\, ds\, dr \le$$

$$\le M \prod_{i=0}^{3} \left(\int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(r,s,t)|^{p_i} \, dt\, ds\, dr \right),$$

where

$$\text{(2.9)} \quad M = (1/8)^{p_0} (1/4)^{p_1+p_2+p_3} [(k-a)(m-b)(n-c)]^{p_0-1} \times$$

$$\times [(m-b)(n-c)]^{p_1-1} [(k-a)(n-c)]^{p_2-1} [(k-a)(m-b)]^{p_3-1}.$$

Theorem 4. Let $1 < p_i < \infty$, *for* $i = 0,1,2,3,4$ *be constants and suppose that* $f \in F(\Delta)$. *Then*

$$\text{(2.10)} \quad \int_a^k \int_b^m \int_c^n |f(r,s,t)|^{p_0} |D_1 f(r,s,t)|^{p_1} |D_2 f(r,s,t)|^{p_2} |D_3 f(r,s,t)|^{p_3} \times$$

$$\times |D_3 D_2 D_1 f(r,s,t)|^{p_4} \, dt\, ds\, dr \le$$

$$\le M M_0 \prod_{i=0}^{4} \left(\int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(r,s,t)|^{2p_i} \, dt\, ds\, dr \right)^{1/2},$$

where M is defined by (2.9) *and*

$$\text{(2.11)} \quad M_0 = [(k-a)(m-b)(n-c)]^{3/2}.$$

Remark 3. We note that the inequalities established in Theorem 3 and 4 are motivated by the inequalities established by Pachpatte in [9] and [10] for functions of two independent variables. By taking $p_i = 1$ for $i = 0,1,2,3,4$ in Theorems 3 and 4 we get the inequalities of Opial type (see [6]) in three independent variables, which we believe are also new to the literature.

3. Proofs of Theorems 1 and 2

From the hypotheses of Theorem 1, it is easy to observe that the following identities hold for $(r,s,t)\in\Delta$,

$$f(r,s,t)=\int_a^r\int_b^s\int_c^t D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.1}$$

$$f(r,s,t)=-\int_a^r\int_b^s\int_t^n D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.2}$$

$$f(r,s,t)=-\int_a^r\int_s^m\int_c^t D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.3}$$

$$f(r,s,t)=-\int_r^k\int_b^s\int_c^t D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.4}$$

$$f(r,s,t)=\int_a^r\int_s^m\int_t^n D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.5}$$

$$f(r,s,t)=\int_r^k\int_s^m\int_c^n D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.6}$$

$$f(r,s,t)=\int_r^k\int_b^s\int_c^n D_3D_2D_1f(u,v,w)dwdvdu, \tag{3.7}$$

$$f(r,s,t)=-\int_r^k\int_s^m\int_t^n D_3D_2D_1f(u,v,w)dwdvdu. \tag{3.8}$$

From (3.1) – (3.8), it is easy to observe that

$$|f(r,s,t)|\le\frac{1}{8}\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(u,v,w)|dwdvdu, \tag{3.9}$$

for $(r,s,t)\in\Delta$. Similarly, we obtain

$$|g(r,s,t)|\le\frac{1}{8}\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(u,v,w)|dwdvdu, \tag{3.10}$$

for $(r,s,t)\in\Delta$. From (3.9), (3.10) and using the Schwarz inequality we get

(3.11)
$$|f(r,s,t)|^2 \le \left(\frac{1}{8}\right)^2 [(k-a)(m-b)(n-c)] \times$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(u,v,w)|^2 \, dwdvdu,$$

(3.12)
$$|g(r,s,t)|^2 \le \left(\frac{1}{8}\right)^2 [(k-a)(m-b)(n-c)] \times$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 g(u,v,w)|^2 \, dwdvdu.$$

From (3.9), (3.10) and using the elementary inequality $a_1 a_2 \le (1/2)(a_1^2 + a_2^2)$ (for a_1, a_2 reals) and (3.11), (3.12) we observe that

$$\int_a^k \int_b^m \int_c^n |f(r,s,t)||g(r,s,t)| \, dtdsdr \le$$
$$\le \frac{1}{2} \int_a^k \int_b^m \int_c^n \left[|f(r,s,t)|^2 + |g(r,s,t)|^2 \right] dtdsdr \le$$
$$\le \frac{1}{2} \int_a^k \int_b^m \int_c^n \Big[(1/8)^2 [(k-a)(m-b)(n-c)] \times$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(u,v,w)|^2 \, dw\, dv\, du +$$
$$+ (1/8)^2 [(k-a)(m-b)(n-c)] \times$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 g(u,v,w)|^2 \, dwdvdu \Big] \, dtdsdr =$$
$$= \frac{1}{2} \left[\frac{(k-a)(m-b)(n-c)}{8} \right]^2 \int_a^k \int_b^m \int_c^n \Big[|D_3 D_2 D_1 f(r,s,t)|^2 +$$
$$+ |D_3 D_2 D_1 g(r,s,t)|^2 \Big] \, dtdsdr.$$

This is the required inequality (2.1) in Theorem 1.

By using Schwarz inequality, the inequality (2.3) to the function f and g, and the elementary inequality $a_1^{1/2}a_2^{1/2} \le (1/2)(a_1 + a_2)$, (for a_1, a_2 nonnegative reals) we observe that

$$\int_a^k\int_b^m\int_c^n \left[|f(r,s,t)||D_3D_2D_1g(r,s,t)| + |g(r,s,t)||D_3D_2D_1g(r,s,t)|\right]dtdsdr \le$$

$$\le \left[\int_a^k\int_b^m\int_c^n |f(r,s,t)|^2\,dtdsdr\right]^{1/2}\left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1g(r,s,t)|^2\,dtdsdr\right]^{1/2} +$$

$$+ \left[\int_a^k\int_b^m\int_c^n |g(r,s,t)|^2\,dtdsdr\right]^{1/2}\left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^2\,dtdsdr\right]^{1/2} \le$$

$$\le \frac{(k-a)(m-b)(n-c)}{8}\left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^2\,dtdsdr\right]^{1/2} \times$$

$$\times \left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1g(r,s,t)|^2\,dtdsdr\right]^{1/2} +$$

$$+ \frac{(k-a)(m-b)(n-c)}{8}\left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1g(r,s,t)|^2\,dtdsdr\right]^{1/2} \times$$

$$\times \left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^2\,dtdsdr\right]^{1/2} =$$

$$= 2\left[\frac{(k-a)(m-b)(n-c)}{8}\right]\left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^2\,dtdsdr\right]^{1/2} \times$$

$$\times \left[\int_a^k\int_b^m\int_c^n |D_3D_2D_1g(r,s,t)|^2\,dtdsdr\right]^{1/2} =$$

$$= \frac{(k-a)(m-b)(n-c)}{8}\int_a^k\int_b^m\int_c^n \Big[|D_3D_2D_1f(r,s,t)|^2 +$$

$$+ |D_3D_2D_1g(r,s,t)|^2\Big]\,dtdsdr.$$

This is the required inequality in (2.2) and the proof of Theorem 1 is complete.

From the hypotheses of Theorem 2, it is easy to observe that (3.9) and (3.10) hold. Similarly, we obtain

$$|h(r,s,t)| \leq \frac{1}{8}\int_a^k\int_b^m\int_c^n |D_3D_2D_1h(u,v,w)|dwdvdu, \tag{3.13}$$

for $(r,s,t)\in\Delta$. From (3.9), (3.10) and (3.13) and using the Hölder's inequality with indices p_1, $p_1/(p_1-1)$ and p_2, $p_2/(p_2-1)$ and p_3, $p_3/(p_3-1)$ repectively we have

$$|f(r,s,t)|^{p_1} \leq \left(\frac{1}{8}\right)^{p_1}[(k-a)(m-b)(n-c)]^{p_1-1}\times \tag{3.14}$$
$$\times\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(u,v,w)|^{p_1}\,dwdvdu,$$

and

$$|g(r,s,t)|^{p_2} \leq \left(\frac{1}{8}\right)^{p_2}[(k-a)(m-b)(n-c)]^{p_2-1}\times \tag{3.15}$$
$$\times\int_a^k\int_b^m\int_c^n |D_3D_2D_1g(u,v,w)|^{p_2}\,dwdvdu,$$

and

$$|h(r,s,t)|^{p_3} \leq \left(\frac{1}{8}\right)^{p_3}[(k-a)(m-b)(n-c)]^{p_3-1}\times \tag{3.16}$$
$$\times\int_a^k\int_b^m\int_c^n |D_3D_2D_1h(u,v,w)|^{p_3}\,dwdvdu.$$

From (3.14) – (3.16) and using the elementary inequality $b_1b_2+b_2b_3+b_3b_1\leq$ $\leq b_1^2+b_2^2+b_3^2$ (for b_1, b_2, b_3 reals) and the repeated application of Schwarz inequality we obtain

$$|f(r,s,t)|^{p_1}|g(r,s,t)|^{p_2}+|g(r,s,t)|^{p_2}|h(r,s,t)|^{p_3}+ \tag{3.17}$$
$$+|h(r,s,t)|^{p_3}|f(r,s,t)|^{p_1}\leq$$
$$\leq\left[|f(r,s,t)|^{p_1}\right]^2+\left[|g(r,s,t)|^{p_2}\right]^2+\left[|h(r,s,t)|^{p_3}\right]^2\leq$$

$$\leq \left[\left(\frac{1}{8}\right)^{p_1} \left[(k-a)(m-b)(n-c)\right]^{p_1-1} \times \right.$$

$$\left. \times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 f(u,v,w) \right|^{p_1} dw dv du \right]^2 +$$

$$+ \left[\left(\frac{1}{8}\right)^{p_2} \left[(k-a)(m-b)(n-c)\right]^{p_2-1} \times \right.$$

$$\left. \times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 g(u,v,w) \right|^{p_2} dw dv du \right]^2 +$$

$$+ \left[\left(\frac{1}{8}\right)^{p_3} \left[(k-a)(m-b)(n-c)\right]^{p_3-1} \times \right.$$

$$\left. \times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 h(u,v,w) \right|^{p_3} dw dv du \right]^2 \leq$$

$$\leq \left(\frac{1}{8}\right)^{2p_1} \left[(k-a)(m-b)(n-c)\right]^{2(p_1-1)} \left[(k-a)(m-b)(n-c)\right] \times$$

$$\times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 f(u,v,w) \right|^{2p_1} dw dv du +$$

$$+ \left(\frac{1}{8}\right)^{2p_2} \left[(k-a)(m-b)(n-c)\right]^{2(p_2-1)} \left[(k-a)(m-b)(n-c)\right] \times$$

$$\times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 g(u,v,w) \right|^{2p_2} dw dv du +$$

$$+ \left(\frac{1}{8}\right)^{2p_3} \left[(k-a)(m-b)(n-c)\right]^{2(p_3-1)} \left[(k-a)(m-b)(n-c)\right] \times$$

$$\times \int_a^k \int_b^m \int_c^n \left| D_3 D_2 D_1 h(u,v,w) \right|^{2p_3} dw dv du.$$

Integeating both sides of (3.17) on Δ we get the desired inequality (2.5) in Theorem 2.

From (3.14) – (3.16) and using the elementary inequalities $c_1c_2c_3(c_1+c_2+ +c_3) \le (1/3)(c_1c_2+c_2c_3+c_3c_1)^2$, $c_1c_2+c_2c_3+c_3c_1 \le c_1^2+c_2^2+c_3^2$, $(c_1+c_2+ +c_3)^2 \le 3(c_1+c_2+c_3)^2$, (for c_1,c_2,c_3 reals) and repeated application of Schwarz inequality we observe that

(3.18)
$$|f(r,s,t)|^{p_1}|g(r,s,t)|^{p_2}|h(r,s,t)|^{p_3}\times$$
$$\times\left[|f(r,s,t)|^{p_1}+|g(r,s,t)|^{p_2}+|h(r,s,t)|^{p_3}\right]\le$$
$$\le\left[\{|f(r,s,t)|^{p_1}\}^2\right]^2+\left[\{|g(r,s,t)|^{p_2}\}^2\right]^2+\left[\{|h(r,s,t)|^{p_3}\}^2\right]^2\le$$
$$\le\left[\left[\left(\frac{1}{8}\right)^{p_1}[(k-a)(m-b)(n-c)]^{p_1-1}\times\right.\right.$$
$$\left.\left.\times\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(u,v,w)|^{p_1}\,dwdvdu\right]^2\right]^2+$$
$$+\left[\left[\left(\frac{1}{8}\right)^{p_2}[(k-a)(m-b)(n-c)]^{p_2-1}\times\right.\right.$$
$$\left.\left.\times\int_a^k\int_b^m\int_c^n|D_3D_2D_1g(u,v,w)|^{p_2}\,dwdvdu\right]^2\right]^2+$$
$$+\left[\left[\left(\frac{1}{8}\right)^{p_3}[(k-a)(m-b)(n-c)]^{p_3-1}\times\right.\right.$$
$$\left.\left.\times\int_a^k\int_b^m\int_c^n|D_3D_2D_1h(u,v,w)|^{p_3}\,dwdvdu\right]^2\right]^2\le$$
$$\le\left(\frac{1}{8}\right)^{4p_1}[(k-a)(m-b)(n-c)]^{4(p_1-1)}[(k-a)(m-b)(n-c)]^2\times$$
$$\times\left[\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(u,v,w)|^{2p_1}\,dwdvdu\right]^2+$$

$$+\left(\frac{1}{8}\right)^{4p_2}\left[(k-a)(m-b)(n-c)\right]^{4(p_2-1)}\left[(k-a)(m-b)(n-c)\right]^2\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n\left|D_3D_2D_1g(u,v,w)\right|^{2p_2}dwdvdu\right]^2+$$

$$+\left(\frac{1}{8}\right)^{4p_3}\left[(k-a)(m-b)(n-c)\right]^{4(p_3-1)}\left[(k-a)(m-b)(n-c)\right]^2\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n\left|D_3D_2D_1f(u,v,w)\right|^{2p_3}dwdvdu\right]^2\leq$$

$$\leq\left(\frac{1}{8}\right)^{4p_1}\left[(k-a)(m-b)(n-c)\right]^{4(p_1-1)}\left[(k-a)(m-b)(n-c)\right]^2\times$$

$$\times\left[(k-a)(m-b)(n-c)\right]\int_a^k\int_b^m\int_c^n\left|D_3D_2D_1f(u,v,w)\right|^{4p_1}dwdvdu+$$

$$+\left(\frac{1}{8}\right)^{4p_2}\left[(k-a)(m-b)(n-c)\right]^{4(p_2-1)}\left[(k-a)(m-b)(n-c)\right]^2\times$$

$$\times\left[(k-a)(m-b)(n-c)\right]\int_a^k\int_b^m\int_c^n\left|D_3D_2D_1g(u,v,w)\right|^{4p_2}dwdvdu+$$

$$+\left(\frac{1}{8}\right)^{4p_3}\left[(k-a)(m-b)(n-c)\right]^{4(p_3-1)}\left[(k-a)(m-b)(n-c)\right]^2\times$$

$$\times\left[(k-a)(m-b)(n-c)\right]\int_a^k\int_b^m\int_c^n\left|D_3D_2D_1h(u,v,w)\right|^{4p_3}dwdvdu.$$

Integrating both sides of (3.18) on Δ we get the desired inequality in (2.6). The proof of the Theorem 2 is complete.

4. PROOFS OF THEOREMS 3 AND 4

From the hypotheses of Theorem 3, it is easy to observe that (3.1) – (3.9) hold and also the following identilies hold for $(r,s,t)\in\Delta$, (4.1) $$D_1 f(r,s,t) = \int_b^s \int_c^t D_3 D_2 D_1 f(r,v,w)\,dw\,dv,$$

(4.2) $$D_1 f(r,s,t) = -\int_b^s \int_t^n D_3 D_2 D_1 f(r,v,w)\,dw\,dv,$$

(4.3) $$D_1 f(r,s,t) = -\int_s^m \int_c^t D_3 D_2 D_1 f(r,v,w)\,dw\,dv,$$

(4.4) $$D_1 f(r,s,t) = \int_s^m \int_t^n D_3 D_2 D_1 f(r,v,w)\,dw\,dv,$$

and

(4.5) $$D_2 f(r,s,t) = \int_a^r \int_c^t D_3 D_2 D_1 f(u,s,w)\,dw\,du,$$

(4.6) $$D_2 f(r,s,t) = -\int_a^r \int_t^n D_3 D_2 D_1 f(u,s,w)\,dw\,du,$$

(4.7) $$D_2 f(r,s,t) = -\int_r^k \int_c^t D_3 D_2 D_1 f(u,s,w)\,dw\,du,$$

(4.8) $$D_2 f(r,s,t) = \int_r^k \int_t^n D_3 D_2 D_1 f(u,s,w)\,dw\,du,$$

and

(4.9) $$D_3 f(r,s,t) = \int_a^r \int_b^s D_3 D_2 D_1 f(u,v,t)\,dv\,du,$$

(4.10) $$D_3 f(r,s,t) = -\int_a^r \int_s^m D_3 D_2 D_1 f(u,v,t)\,dv\,du,$$

(4.11) $$D_3 f(r,s,t) = -\int_r^k \int_b^s D_3 D_2 D_1 f(u,v,t)\,dv\,du,$$

(4.12) $$D_3 f(r,s,t) = \int_r^k \int_s^m D_3 D_2 D_1 f(u,v,t)\,dv\,du.$$

Form (4.1) – (4.4), (4.5) – (4.8), (4.9) – (4.12), it is easy to observe that

$$|D_1 f(r,s,t)| = \frac{1}{4}\int_b^m \int_c^n |D_3 D_2 D_1 f(r,v,w)| dw dv, \tag{4.13}$$

$$|D_2 f(r,s,t)| = \frac{1}{4}\int_a^k \int_c^n |D_3 D_2 D_1 f(u,s,w)| dw du, \tag{4.14}$$

$$|D_3 f(r,s,t)| = \frac{1}{4}\int_a^k \int_b^m |D_3 D_2 D_1 f(u,v,t)| dv du. \tag{4.15}$$

By taking the powers p_0, p_1, p_2, p_3 respectively on both sides of (3.9), (4.13), (4.14), (4.15) and then using the Hölder's inequality with indices p_0, $p_0/(p_0-1)$ and p_1, $p_1/(p_1-1)$ and p_2, $p_2/(p_2-1)$ and p_3, $p_3/(p_3-1)$ respectively on the right sides we get

$$|f(r,s,t)|^{p_0} = \left(\frac{1}{8}\right)^{p_0} [(k-a)(m-b)(n-c)]^{p_0-1} \times \tag{4.16}$$
$$\times \int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(u,v,w)|^{p_0} dw dv du,$$

$$|D_1 f(r,s,t)|^{p_1} = \left(\frac{1}{4}\right)^{p_1} [(m-b)(n-c)]^{p_1-1} \times \tag{4.17}$$
$$\times \int_b^m \int_c^n |D_3 D_2 D_1 f(r,v,w)|^{p_1} dw dv,$$

$$|D_2 f(r,s,t)|^{p_2} = \left(\frac{1}{4}\right)^{p_2} [(k-a)(n-c)]^{p_2-1} \times \tag{4.18}$$
$$\times \int_a^k \int_c^n |D_3 D_2 D_1 f(u,s,w)|^{p_2} dw du,$$

$$|D_3 f(r,s,t)|^{p_3} = \left(\frac{1}{4}\right)^{p_3} [(k-a)(m-b)]^{p_3-1} \times \tag{4.19}$$
$$\times \int_a^k \int_b^m |D_3 D_2 D_1 f(u,v,t)|^{p_3} dv du,$$

From (4.16) – (4.19) we observe that

$$(3.18)\quad |f(r,s,t)|^{p_0}|D_1f(r,s,t)|^{p_1}|D_2f(r,s,t)|^{p_2}|D_3f(r,s,t)|^{p_3}\le$$

$$\le\left[\left(\frac{1}{8}\right)^{p_0}[(k-a)(m-b)(n-c)]^{p_0-1}\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(u,v,w)|^{p_0}\,dwdvdu\right]\times$$

$$\times\left[\left(\frac{1}{4}\right)^{p_1}[(m-b)(n-c)]^{p_1-1}\int_b^m\int_c^n|D_3D_2D_1f(r,v,w)|^{p_1}\,dwdv\right]\times$$

$$\times\left[\left(\frac{1}{4}\right)^{p_2}[(k-a)(n-c)]^{p_2-1}\int_a^k\int_c^n|D_3D_2D_1f(u,s,w)|^{p_2}\,dwdu\right]\times$$

$$\times\left[\left(\frac{1}{4}\right)^{p_3}[(k-a)(m-b)]^{p_3-1}\int_a^k\int_b^m|D_3D_2D_1f(u,v,t)|^{p_3}\,dvdu\right].$$

Now integrating both sides of (4.20) on Δ and using Fubini's theorem we get the required inequality in (2.8). This completes the proof of Theorem 3.

From the hypotheses of Theorem 4, it is easy to see that (4.16) – (4.20) hold. Multiplying both sides of (4.20) by $|D_3D_2D_1f(r,s,t)|^{p_4}$, integrating the resulting inequality on Δ and using the Schwarz inequality repeatedly and then Fubini's theorem we observe that

$$\int_a^k\int_b^m\int_c^n|f(r,s,t)|^{p_0}|D_1f(r,s,t)|^{p_1}|D_2f(r,s,t)|^{p_2}\times$$

$$\times|D_3f(r,s,t)|^{p_3}|D_3D_2D_1f(r,s,t)|^{p_4}\,dtdsdr\le$$

$$\le M\left(\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{p_0}\,dtdsdr\right)\times$$

$$\times\int_a^k\int_b^m\int_c^n\left[\left(\int_b^m\int_c^n|D_3D_2D_1f(r,v,w)|^{p_1}\,dwdv\right)\left(\int_a^k\int_c^n|D_3D_2D_1f(u,s,w)|^{p_2}\,dwdu\right)\times\right.$$

$$\left.\times\left(\int_a^k\int_b^m|D_3D_2D_1f(u,v,t)|^{p_3}\,dvdu\right)|D_3D_2D_1f(r,s,t)|^{p_4}\right]dtdsdr\le$$

$$\le M\left(\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{p_0}\,dtdsdr\right)\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n\left(\int_b^m\int_c^n|D_3D_2D_1f(r,v,w)|^{p_1}\,dwdv\right)^2\times\right.$$

$$\times\left(\int_a^k\int_c^n|D_3D_2D_1f(u,s,w)|^{p_2}\,dwdu\right)^2\times$$

$$\left.\times\left(\int_a^k\int_b^m|D_3D_2D_1f(u,s,w)|^{p_3}\,dvdu\right)^2 dtdsdr\right]^{1/2}\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{2p_4}\,dtdsdr\right]^{1/2}\leq$$

$$\leq M\left[(k-a)(m-b)(n-c)\right]^{1/2}\left[\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{2p_0}\,dtdsdr\right]^{1/2}\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n\left([(m-b)(n-c)]\int_b^m\int_c^n|D_3D_2D_1f(r,v,w)|^{2p_1}\,dwdv\right)\times\right.$$

$$\times\left([(k-a)(n-c)]\int_a^k\int_c^n|D_3D_2D_1f(u,s,w)|^{2p_2}\,dwdu\right)\times$$

$$\left.\times\left([(k-a)(m-b)]\int_a^k\int_b^m|D_3D_2D_1f(u,v,t)|^{2p_3}\,dvdu\right)dtdsdr\right]^{1/2}\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{2p_4}\,dtdsdr\right]^{1/2}=$$

$$=M\,M_0\left(\int_a^k\int_b^m\int_c^n|D_3D_2D_1f(r,s,t)|^{2p_0}\,dtdsdr\right)\times$$

$$\times\left[\int_a^k\int_b^m\int_c^n\left(\int_b^m\int_c^n|D_3D_2D_1f(r,v,w)|^{2p_1}\,dwdv\right)\times\right.$$

$$\times\left(\int_a^k\int_c^n |D_3D_2D_1f(u,s,w)|^{2p_2}\,dwdu\right)\times$$

$$\times\left(\int_a^k\int_b^m |D_3D_2D_1f(u,v,t)|^{2p_3}\,dvdu\right)dtdsdr\Bigg]^{1/2}\times$$

$$\times\left(\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^{2p_4}\,dtdsdr\right)^{1/2}=$$

$$=MM_0\prod_{i=0}^{4}\left(\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|^{2p_i}\,dtdsdr\right)^{1/2}.$$

This completes the proof of Theorem 4.

Remark 3. The multidimensional integral inequalities of the Wirtinger and Opial type are established by many authors by using different techniques, see [1, 5]. Here we note that our results are obtained by using quite elementary method and we believe that the inequalities established in this paper are of independent interest.

We also note that the special versions of the inequalities given here can be used to study the problems of uniqueness of solutions of certain partial differential equations by modifying the applications given in [1, pp. 360-362]. Here we do not discuss the details.

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