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GROWTH ESTIMATES FOR SOLUTIONS OF DIFFERENCE EQUATIONS

ABSTRACT: In the paper asymptotic estimates for nonoscillatory solutions of some kinds of difference equations are given.

KEY WORDS: difference equation, asymptotic behaviour.

The asymptotic behaviour of the solutions of difference equations of second order were considered by e.g. A. Drozdowicz and J. Popenda [1, 2], J. Hooker and W. Patula [4], W. Patula [6], J. Popenda and J. Werbowski [7], while for $n-$th order difference equations by J. Korczak and M. Migda [5]. In this paper we consider equations which are somewhat similar to these considered in [2]. The methods we use are partly, with of course suitable modifications, similar to these used for differential equations by H. Gollwitzer in [3].

NOTATIONS AND PRELIMINARY RESULTS

In the paper we will use the following notations: $N=\{1,2,3,...\}$ – set of positive integers, $R=(-\infty,\infty)$, $R_0=[0,\infty)$, $R_+=(0,\infty)$, $R_-=(-\infty,0)$ set of real numbers, nonnegative, positive real numbers respectively.

For the function $y:N\to R$ the difference operator Δ is defined as follows

$$\Delta y_n = y_{n+1}-y_n, \qquad n\in N,$$

and

$$\Delta^2 y_n = y_{n+2}-2y_{n+1}+y_n.$$

We use also the convenient assumption that the void sum is equal zero. We will say that the nonnegative sequence $\{y_n\}_{n=1}^{\infty}$ satisfies condition $(*)$ if

$$(*) \qquad \sup_{k\ge n} y_k > 0 \quad \text{for all} \quad n\in N.$$

We start with two lemmas which are essential in our next considerations.

Lemma 1. Let $y=\{y_n\}_{n=1}^{\infty}$ *be positive, increasing sequence such that*

$$(1) \qquad \Delta^2 y_n + p_n y_n^{\alpha} \le 0 \quad \textit{for all} \quad n\ge n_1,$$

where $n_1 \in N$, $p: N \to R_0$, $\alpha > 1$ *is any constant. Then there exists a constant* C *such that*

$$\text{(2)} \qquad \left[\sum_{j=n}^{\infty} j\, p_j\right] y_n^{\alpha-1} \le C \qquad \text{for all } n \ge n_1.$$

Proof. Multiplying inequality (1) by $n y_n^{-\alpha}$ we get

$$n y_n^{-\alpha} \Delta^2 y_n + n p_n \le 0 \qquad n \ge n_1.$$

From the above inequality, by summation, we obtain

$$\sum_{j=m}^{n} j\, y_j^{-\alpha} \Delta^2 y_j + \sum_{j=m}^{n} j\, p_j \le 0 \qquad \text{for } n \ge m \ge n_1.$$

Applying the method summation by parts we can transform this inequality to the from

$$\text{(3)} \qquad n y_n^{-\alpha} \Delta y_{n+1} - m y_m^{-\alpha} \Delta y_m - \sum_{j=m+1}^{n} (\Delta((j-1) y_{j-1}^{-\alpha}) \Delta y_j) + \sum_{j=m}^{n} j\, p_j \le 0.$$

Since $n y_n^{-\alpha} \Delta y_{n+1}$ is positive and

$$\Delta[(j-1) y_{j-1}^{-\alpha}] = y_j^{-\alpha} + (j-1)\Delta(y_{j-1}^{-\alpha})$$

then from (3) we get

$$\text{(4)} \qquad \sum_{j=m}^{n} j p_j < m y_m^{-\alpha} \Delta y_m + \sum_{j=m+1}^{n} y_j^{-\alpha} \Delta y_j + \sum_{j=m+1}^{n} (j-1)\Delta(y_{j-1}^{-\alpha}) \Delta y_j.$$

The sequence $\{y_n^{-\alpha}\}_{n=1}^{\infty}$ is decreasing for $n \ge n_1$, therefore $\Delta(y_n^{-\alpha}) < 0$ for $n \ge n_1$, while $\Delta y_n > 0$, so the third term on the right hand side of (4) is negative. Furthermore from (1) it follows that $\Delta y_j \le \Delta y_{j-1}$ for $j > n_1$. Therefore (4) yields

$$\text{(5)} \qquad \sum_{j=m}^{n} j p_j < m y_m^{-\alpha} \Delta y_m + \sum_{j=m+1}^{n} y_j^{-\alpha} \Delta y_{j-1}.$$

Considering the function $f(x) = x^{-\alpha}$ one can easy check that

$$\text{(6)} \qquad y_j^{-\alpha} \Delta y_{j-1} < \int_{y_{j-1}}^{y_j} \frac{1}{s^{\alpha}}\, ds = \frac{y_j^{-\alpha+1}}{-\alpha+1} - \frac{y_{j-1}^{-\alpha+1}}{-\alpha+1}, \qquad j > n_1.$$

Using (6) in (5) we have

$$\sum_{j=m}^{n} j\,p_j < my_m^{-\alpha}\Delta y_m + \frac{y_n^{-\alpha+1}}{-\alpha+1} - \frac{y_m^{-\alpha+1}}{-\alpha+1} < my_m^{-\alpha}\Delta y_m + \frac{y_m^{-\alpha+1}}{\alpha-1}.$$

This inequality holds for all $n \geq m$ so

$$\sum_{j=m}^{n} j\,p_j < my_m^{-\alpha}\Delta y_m + \frac{y_m^{-\alpha+1}}{\alpha-1}$$

and in consequence we get

$$\text{(7)} \qquad \left[\sum_{j=m}^{n} j\,p_j\right] y_m^{\alpha-1} \leq m\,y_m^{-1}\,\Delta y_m + (\alpha-1)^{-1}, \qquad m \geq n_1.$$

Notice that for any sequence y which fulfils inequality (1) there is

$$\Delta[n\,\Delta y_n - y_n] = (n+1)\,\Delta^2 y_n \leq -(n+1)\,p_n y_n^{\alpha} \leq 0, \qquad \text{for} \quad n \geq n_1$$

and this means that the sequence $\{n\,\Delta y - y_n\}_{n=n_1}^{\infty}$ is non-increasing. Therefore

$$n\,\Delta y_n - y_n \leq n_1 \Delta y_{n_1} - y_{n_1} \leq n_1 \Delta y_{n_1} \qquad \text{for all} \quad n \geq n_1.$$

From the above, dividing by y_n we obtain

$$\text{(8)} \qquad n\,y_n^{-1}\Delta y_n \leq 1 + n_1 y_n^{-1}\Delta y_{n_1} \leq 1 + n_1 y_{n_1}^{-1}\Delta y_{n_1}, \qquad n \geq n_1,$$

because the sequence $\{y_n^{-1}\}$ is strictly decreasing for $n \geq n_1$. Denoting the sum $1 + n_1 y_{n_1}^{-1}\Delta y_{n_1}$ by C_1 and applying estimation (8) in (7) we finally get

$$\left[\sum_{j=m}^{\infty} j\,p_j\right] y_m^{\alpha-1} \leq C_1 + (\alpha-1)^{-1}, \qquad m \geq n_1,$$

that is inequality (2) with $C = C_1 + (\alpha-1)^{-1}$.

Lemma 2. Let $y = \{y_n\}_{n=1}^{\infty}$ *be positive, increasing sequence satisfying inequality* (1) *where* $p\colon N \to R_0$, $\alpha \in (0,1)$ *is any constant. Then there exists a constant C such that*

$$\text{(9)} \qquad n^{1-\alpha}\left[\sum_{j=m}^{\infty} j^{\alpha} p_j\right] y_n^{\alpha-1} \leq C \qquad \textit{for all} \quad n \geq n_1.$$

Proof. Multiplying inequality (1) by positive term $(\Delta y_n)^{-\alpha}$ we obtain

(10) $$(\Delta y_n)^{-\alpha}\Delta^2 y_n + p_n y_n^{\alpha}(\Delta y_n)^{-\alpha} \le 0, \qquad n \ge n_1.$$

The assumptions of the lemma assure likewise in Lemma 1 the bound

$$n y_n^{-1}\Delta y_n \le C_1,$$

where $C_1 = 1 + n_1 y_{n_1}^{-1}\Delta y_{n_1},$ from there we can easy deduce that

(11) $$[y_n(\Delta y_n)^{-1}]^{\alpha} \ge C_1^{-\alpha} n^{\alpha} \qquad \text{for } n \ge n_1.$$

Applying (11) in (10) we have

$$(\Delta y_n)^{-\alpha}\Delta^2 y_n + p_n C_1^{-\alpha} n^{\alpha} \le 0 \qquad \text{for } n \ge n_1.$$

So, by summation, we get

(12) $$\sum_{j=m}^{n}[\Delta y_j]^{-\alpha}\Delta^2 y_j + C_1^{-\alpha}\sum_{j=m}^{n} j^{\alpha} p_j \le 0 \qquad \text{for } n \ge n_1.$$

A similar reasoning as in the proof of Lemma 1, for the function $f(x) = x^{-\alpha}$ but now for $0 < \alpha < 1$, allows us to obtain the following estimate

$$(\Delta y_j)^{-\alpha}\Delta^2 y_j < \int_{\Delta y_j}^{\Delta y_{j+1}} \frac{ds}{s^{\alpha}} = \frac{(\Delta y_{j+1})^{1-\alpha}}{-\alpha+1} - \frac{(\Delta y_j)^{1-\alpha}}{-\alpha+1}.$$

Taking this bound into account, we state by (12) that

$$\frac{(\Delta y_{n+1})^{1-\alpha}}{1-\alpha} - \frac{(\Delta y_m)^{1-\alpha}}{1-\alpha} + C_1^{-\alpha}\sum_{j=m}^{n} j^{\alpha} p_j \le 0.$$

The first term on the left hand side of the above inequality is positive therefore

$$\sum_{j=m}^{n} j^{\alpha} p_j \le C_1^{\alpha}(1-\alpha)^{-1}(\Delta y_m)^{1-\alpha}.$$

This bound remains true if n tends to infinity, and as result of $1-\alpha > 0$ and because both terms are positive we have

(13) $$\left[\sum_{j=m}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)} \le C_2 \Delta y_m, \qquad m \ge n_1,$$

where

$$C_2 = C_1^{\alpha/(1-\alpha)}(1-\alpha)^{-1/(1-\alpha)}.$$

Summing (13) from n_1 to $n-1$ we obtain

$$\sum_{m=n_1}^{n-1}\left[\sum_{j=m}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)} \leq C_2 y_n - C_2 y_{n_1} < C_2 y_n \qquad \text{for } n \geq n_1. \tag{14}$$

For $n \geq 2n_1$ we have

$$\sum_{m=n_1}^{n-1}\left[\sum_{j=m}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)} \leq (n-n_1)\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)} \leq \frac{n}{2}\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)}.$$

Therefore, by (14)

$$n\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right]^{1/(1-\alpha)} \leq 2C_2 y_n \qquad \text{for } n \geq 2n_1,$$

from there

$$n^{1-\alpha}\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right] \leq C_3 y_n^{1-\alpha} \qquad \text{for } n \geq 2n_1,$$

where $C_3 = (2C_2)^{1-\alpha}$. Hence

$$n^{1-\alpha}\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right] y_n^{\alpha-1} \leq C_3 \qquad \text{for } n \geq 2n_1.$$

That is (9) holds for $n \geq 2n_1$. Since $2n_1$ is finite we can find such a constant C_4 that this estimate remains true also for $n \in \{n_1, \ldots, 2n_1 - 1\}$. The lemma is proved.

MAIN RESULTS

Let the sequence $y = \{y_n\}_{n=1}^{\infty}$ be the solution of any difference equation. We call this solution trivial or zero solution if there exists $n_1 \in N$ such that $y_n = 0$ for all $n \geq n_1$, otherwise this solution is called nontrivial. The nontrivial solution is oscillatory if for all $m \in N$ there exists $n \geq m$ such that $y_n y_{n+1} \leq 0$. Otherwise it is called nonoscillatory. In the next theorems we consider nonoscillatory solutions.

Theorem 1. Let p be nonnegative sequence on N, α fixed constant $\alpha > 1$. If $y = \{y_n\}_{n=1}^{\infty}$ is a nonoscillatory solution of the equation

(E1) $$\Delta^2 y_n + p_n |y_n|^{\alpha} sgn(y_n) = 0, \qquad n \in N,$$

then there exists a positive constant C such that

(B1) $$\left[\sum_{j=n}^{\infty} j\, p_j\right] |y_n|^{\alpha-1} \le C \qquad \textit{for all } n \in N.$$

Proof. We start by observing that we can consider only eventually positive solutions. Indeed if $\{z_n\}_{n=1}^{\infty}$ is any eventually negative solution then $\{-z_n\}_{n=1}^{\infty}$ is also a solution which is eventually positive and conversely. If we prove validity of (B1) for all $n \ge n_0$ and some $n_0 \in N$ then it will be possible to find new constant C_1, $C_1 \ge C$ such that the estimate (B1) with the constant C_1 instead of C remains true for all $n \in N$.

So let $\{y_n\}_{n=1}^{\infty}$ be any eventually positive solution say for $n \ge n_1$. For $n \ge n_1$ we obtain from (E1)

$$\Delta^2 y_n + p_n y_n^{\alpha} = 0, \quad n \ge n_1.$$

Hence, $\Delta^2 y_n = -p_n y_n^{\alpha} \le 0$ for $n \ge n_1$, therefore the sequence $\{\Delta y_n\}$ is non-increasing. The inequalities $y_n > 0$ and $\Delta^2 y_n \le 0$ for $n \ge n_1$ imply that either:

(i) $\Delta y_n > 0$ for all $n \ge n_1$ or

(ii) there is $n_2 \ge n_1$ such that $\Delta y_n = 0$ for all $n \ge n_2$.

In the case (i), by Lemma 1, we get (B1) and it is evident that in the case (ii) (B1) holds.

Corollary 1. Let in addition to the assumptions of Theorem 1

(15) $$\liminf_{n\to\infty} n^{\mu(\alpha-1)} \sum_{j=n}^{\infty} j\, p_j > 0 \quad \textit{for some } \mu \in (0,1).$$

If $y = \{y_n\}_{n=1}^{\infty}$ *is nonoscillatory solution of* (E1) *then it can be estimated as follows*

(16) $$|y_n| \le C n^{\mu}$$

for some positive constant C.

Proof. From (15) it follows that there exists a positive constant ε and integer $n_{\varepsilon} \in N$ depending on ε, such that

$$n^{\mu(\alpha-1)}\sum_{j=n}^{\infty} jp_j > \varepsilon$$

for all $n \geq n_\varepsilon$. Hence

$$\sum_{j=n}^{\infty} jp_j > \varepsilon n^{-\mu(\alpha-1)}.$$

By Theorem 1, we have that for any nonoscillatory solution $y = \{y_n\}_{n=1}^{\infty}$

$$|y_n|^{\alpha-1}\sum_{j=n}^{\infty} jp_j > C_1$$

for some constant C_1. Therefore

$$|y_n|^{\alpha-1}\,\varepsilon\, n^{-\mu(\alpha-1)} \leq C_1 \qquad \text{for all } n \geq n_\varepsilon.$$

From there

$$|y_n| \leq (C_1/\varepsilon)^{1/(\alpha-1)} n^{\mu}$$

that is (16) holds for at least all n, $n \geq n_\varepsilon$.

Taking $C = \max\{(C_1/\varepsilon)^{1/(\alpha-1)}, \max\limits_{1\leq i\leq n_\varepsilon}(|y_i|/i^{\mu})\}$ we get the bound given by (16).

In a similar manner, applying Lemma 2 instead of Lemma 1 we can prove the following theorem.

Theorem 2. Let p be nonnegative sequence on N, α fixed constant $0 < \alpha < 1$. If $y = \{y_n\}_{n=1}^{\infty}$ is a nonoscillatory solution of the equation (E1), *then there exists a positive constant C such that for all $n \geq n_1$*

(B2)
$$n^{1-\alpha}\left[\sum_{j=n}^{\infty} j^{\alpha} p_j\right]|y_n|^{\alpha-1} \leq C,$$

where $n_1 = \inf\{n : y_k \neq 0 \text{ for all } k \geq n\}$.

Corollary 2. Let in addition to the assumptions of Theorem 2

$$\liminf_{n\to\infty} n^{(1-\nu)(1-\alpha)}\sum_{j=n}^{\infty} j^{\alpha} p_j > 0 \quad \textit{for some } \nu \in (0,1).$$

Then for every nonoscillatory solution $y = \{y_n\}_{n=1}^{\infty}$ of (E1) *there exists a constant C such that*

(17) $$|y_n| \geq Cn^{\nu}$$

for all $n \geq n_1$, *where* $n_1 = \inf\{n : y_k \neq 0 \text{ for all } k \geq n\}$.

Proof. Much the same as in the Corollary 1 we have

$$\sum_{j=n}^{\infty} j^{\alpha} p_j > \varepsilon n^{-(1-\nu)(1-\alpha)}$$

for some ε and all $n \geq n_\varepsilon$. This together with the bound (B2) leads us to the estimation

$$\varepsilon n^{1-\alpha} n^{-(1-\nu)(1-\alpha)} |y_n|^{\alpha-1} \leq C_1$$

for some positive constant C_1 and $n \geq n_2 := \max\{n_\varepsilon, n_1\}$, From there we obtain

$$|y_n| \geq (\varepsilon/C_1)^{1/(1-\alpha)} n^{-(1-\nu)} n = C_2 n^{\nu}.$$

If $n_\varepsilon > n_1$ then we can take

$$C = \min\left\{C_2, \min_{n_1 \leq i \leq n_\varepsilon} (|y_i|/i^{\nu})\right\}$$

to get (17) for all $n \geq n_1$.

Remark 1. Both the Theorems 1, 2 and Corollaries 1, 2 hold for more general equations e.g.

(E2) $$\Delta^2 y_n + (p_n |y_n|^{\alpha} + q_n |y_n|^{\beta}) \operatorname{sgn}(y_n) = 0, \qquad n \in N,$$

where $p, q : N \to R_0$, $\alpha > 1$, $\beta \in (0,1)$, p, q satisfy condition $(*)$.

This is evident because perhaps non-void set of all nonoscillatory solutions of (E2) possesses symmetry property described in the proof of Theorem 1, and any eventually positive nonoscillatory solution is eventually increasing and satisfies inequalities (1) with p, α and q, β respectively to the assumptions of Lemma 1 or Lemma 2.

This remark allows us to formulate the folowing criterion for oscillation of all solutions of (E2)

Theorem 3. Let p, q *be nonnegative sequences on* N, *satisfying condition* $(*)$, $\alpha > 1$, $\beta \in (0,1)$. *If furthermore*

$$\liminf_{n \to \infty} n^{\mu(\alpha-1)} \sum_{j=n}^{\infty} j p_j > 0$$

and

$$\liminf_{n\to\infty} n^{(1-\nu)(1-\beta)} \sum_{j=n}^{\infty} j^{\beta} q_j > 0,$$

where μ, ν *are some reals such that* $0<\mu<\nu<1$, *then every solution* (E2) *is oscillatory.*

Proof. By Remark 1, Corollary 1 and 2 every nonoscillatory solution satisfies simultaneously

$$|y_n| \le C_1 n^{\mu} \quad \text{and} \quad |y_n| \le C_2 n^{\nu}$$

for some positive constants C_1, C_2 and sufficiently large n, say $n \ge n_1$.

Let $y=\{y_n\}_{n=1}^{\infty}$ be positive solution for $n \ge n_1$, then we have

$$C_2 n^{\nu} \le y_n \le C_1 n^{\mu} \quad \text{for all } n \ge n_1.$$

Hence

$$0 < C_2 \le C_1 n^{\mu-\nu} \quad \text{for all } n \ge n_1.$$

Therefore

$$0 < C_2 \le \lim_{n\to\infty} C_1 n^{\mu-\nu} = 0.$$

This contradiction completes the proof.

Remark 2. Notice that Theorem 1 and 2 give us well known criteria for oscillatory behaviour of all solutions of equations (E1). Namely, if in (E1) $\alpha>1$, $p\colon N \to R_0$, and

$$\sum_{j=1}^{\infty} j p_j = \infty,$$

then every solution of this equation is oscillatory. It is evident because for any nonoscillatory solution we get contradiction $\infty \le C$.

Similarly, if $0<\alpha<1$, $p\colon N \to R_0$, and

$$\sum_{j=1}^{\infty} j^{\alpha} p_j = \infty,$$

then all solutions of (E1) are oscillatory. Furthermore in (E1) if $0<\alpha<1$, $p\colon N \to R_0$,

$$\limsup_{n\to\infty} n^{(1-\alpha)} \sum_{j=1}^{\infty} j^{\alpha} p_j = \infty,$$

then every bounded solutions is oscillatory.

Remark 3. Following our considerations we have made in Remark 1 we can formulate similar results to these given in Theorem 1 or Theorem 2 for various types of equations. Namely, we will investigate equation

$$\text{(E3)} \qquad \Delta_a^2 y_n + (\overline{p}_n |y_n|^{\alpha} + \overline{q}_n |y_n|^{\beta}) \operatorname{sgn}(y_n) = 0, \qquad n \in N,$$

where a is some fixed positive constant, $\alpha > 1$, $\beta \in (0,1)$, and the functions $\overline{p}, \overline{q} : N \to R_0$, satisfy condition $(*)$.

The difference operators Δ_a and Δ_a^2 are defined as follows:

$$\Delta_a y_n = y_{n+1} - a y_n, \quad \Delta_a^2 y_n = y_{n+2} - 2a y_{n+1} + a^2 y_n, \quad n \in N.$$

On using substitution

$$\text{(18)} \qquad y_n = a^n w_n, \quad n \in N$$

we transform equation (E3) to the form (E2) with $w = \{w_n\}_{n=1}^{\infty}$ as a unknown sequence and $p_n = \overline{p}_n a^{n\alpha - n - 2}$, $q_n = \overline{q}_n a^{n\beta - n - 2}$. Since a is positive then, by (18) both $\{y_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ oscillates or nonoscillates simultaneously. Hence from Theorem 1 and 2 we obtain the following estimates of the nonoscillatory solution $\{y_n\}_{n=1}^{\infty}$ of (E3)

$$a^{-n(\alpha-1)} \left[\sum_{j=n}^{\infty} j p_j \, a^{j(\alpha-1)} \right] |y_n|^{\alpha-1} \le C_1, \qquad n \in N$$

and

$$n^{1-\beta} a^{n(1-\beta)} \left[\sum_{j=n}^{\infty} j^{\beta} \overline{q}_j \, a^{j(\beta-1)} \right] |y_n|^{\beta-1} \le C_2, \quad n \ge n_1$$

for some positive constants C_1, C_2.

Let us consider the equation

$$\text{(E4)} \qquad \Delta^2 y_n + p_n f(y_n) = 0.$$

If $p : N \to R_0$, $f : R \to R$ and $f(-x) = -f(x)$, $f > 0$ on R_+, then every eventually positive solution is increasing while every eventually negative solution is decreasing.

If furthermore $f(x) > x^{\alpha}$ with $\alpha > 1$ or $f(x) > x^{\beta}$, $\beta \in (0,1)$ for $x \in R_{+}$, then the estimates (B1) or (B2) (with β instead of α) of nonoscillatory solutions hold respectively.

This reasoning and bound (B1) remains true for the nonoscillatory solutions of the equations

(E5) $$\Delta^2 y_n + p_n |y_{n+k}|^{\alpha} \operatorname{sgn}(y_{n+k}) = 0, \qquad n \in N,$$

(E6) $$\Delta^2 y_n + \sum_{i=0}^{k} p_{i,n} |y_{n+i}|^{\alpha_i} \operatorname{sgn}(y_n) = 0, \qquad n \in N,$$

or even more general equation

(E7) $$\Delta^2 y_n + \sum_{i=0}^{k} p_{i,n} f_i(y_{d_i(n)}) = 0, \qquad n \in N,$$

where k is any fixed positive integer, $\alpha > 1$, $\alpha_i > 0$, and for some $j \in \{0,1,...,k\}$, $\alpha_j > 1$, $f_i : R \to R$, $f_i(-x) = -f_i(x)$, $f_i(x) > 0$ for $x > 0$ and for some $s \in \{0,1,...,k\}$ there exists $\gamma > 1$ such that $f_s(x) > x^{\gamma}$ for $x \in R_{+}$, $p, p_i : N \to R_0$ and satisfy condition (*), $d_i : N \to N$ and $d_i(n) \geq n$ for $n \in N$, $i = 0,1,...,k$.

This follows from the fact that eventually positive solution is increasing and satisfies the relation (we present it here for the equation (E7))

$$0 = \Delta^2 y_n + \sum_{i=0}^{k} p_{i,n} f_i(y_{d_i(n)}) \geq \Delta^2 y_n + p_{s,n} f_s(y_{d_s(n)}) \geq$$

$$\geq \Delta^2 y_n + p_{s,n} (y_{d_s(n)})^{\gamma} \geq \Delta^2 y_n + p_{s,n} (y_n)^{\gamma}.$$

Therefore for this solution Lemma 1 is applicable. (We give here the conditions suitable to the Theorem 1, but it is easy to formulate similar equations and assumptions to get bounds of the type (B2) due to the Theorem 2.)

To prove similar results for the equations which are sometimes called equations with retarded argument, we have to observe that for positive sequence $\{y_n\}_{n=1}^{\infty}$ with $\Delta^2 y_n \leq 0$ we obtain

$$(1/2) y_{n+2} \leq (1/2) y_{n+2} + (1/2) y_n = (1/2) \Delta^2 y_n + y_{n+1} \leq y_{n+1}$$

and consequently

$$y_n \geq (1/2)^k y_{n+k}.$$

Hence for eventually positive $\{y_n\}_{n=1}^{\infty}$ of

(E8) $$\Delta^2 y_n + p_n |y_{n-k}|^\alpha \operatorname{sgn}(y_{n-k}) = 0$$

we have

$$0 = \Delta^2 y_n + p_n y_{n-k}^\alpha \geq \Delta^2 y_n + p_n 2^{-k\alpha} y_n^\alpha$$

that is we obtain inequality (1) with $p_n 2^{-k\alpha}$ instead of p_n. The bound (B1) holds with the constant C enlarged by the fixed factor $2^{k\alpha}$.

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