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STANISŁAW STOIŃSKI

ON COMPACTNESS OF ALMOST PERIODIC FUNCTIONS IN THE LEBESGUE MEASURE

ABSTRACT: The paper gives some compactness criterion for almost periodic functions in the Lebesgue measure and the definition and some property of μ-normal functions.

KEY WORDS: almost periodic function, the Lebesgue measure, compactness.

1. PRELIMINARIES

Let □ be the σ-algebra of subsets of the space R^1 which are measurable in the same of Lebesgue, μ the Lebesgue measure, $\aleph$ the space of □-measurable and finite functions $f : R^1 \to R_n$ where $f = g \Leftrightarrow f(t) = g(t)$ almost everywhere in R^1.

Let us write

$$\tilde{\aleph} = \{f \in \aleph : \underset{(\lambda_n)}{\forall} [(\lambda_n > 0, \lambda_n \to 0 \text{ as } n \to \infty) \Rightarrow$$

$$(\sup_{u \in R} \mu(\{t \in <u, u+1> : \lambda_n \,|\, f(t) \,|\geq 1\}) \to 0)]\}$$

$\tilde{\aleph}$ is a linear set and $\tilde{\aleph} \not\subset \aleph$.

For $f \in \tilde{\aleph}$ we define the functional

$$(1) \qquad |\, f \,| = \sup_{u \in R} \int_u^{u+1} \frac{|\, f(t) \,|}{1 + |\, f(t) \,|} \, dt$$

For every $\eta > 0$ let us put

$$D(\eta; f; g) = \sup_{u \in R} \mu(\{t \in <u, u+1> : |\, f(t) - g(t) \,| \geq \eta\})$$

for $f, g \in \aleph$.

The following theorem is true (see [2]).

Theorem 1. Let $f_n, f_0 \in \aleph$ *for* $n = 1, 2, \ldots$. *Then*

$$(|f_n - f_0| \to 0) \Leftrightarrow \underset{\eta>0}{\forall} \underset{\varepsilon>0}{\forall} \underset{N>0}{\exists} \underset{n>N}{\forall} D(\eta; f_n, f_0) \le \varepsilon.$$

Let us denote by $\widetilde{S}$ the space $\widetilde{\aleph}$ in which the *F*-norm of the form (1) is deffined. Hence $\widetilde{S} = <\widetilde{\aleph}, |\ |>$ is an F^*-space. It is known (see [2]) that the space $\tilde{S}$ is complete.

A set $E \subset R$ is called relatively dense iff there is a positive number ω such that in each open interval $(a, a+\omega)$, $a \in R$, there is at least one element of the set *E*.

Let $f \in \aleph$. If for $\eta > 0$, $\varepsilon > 0$ there is

$$D(\eta; f, f_\tau) \le \varepsilon,$$

where $f_\tau(t) = f(t+\tau)$, then the number $\tau \in R$ is called an (ε, η)-almost period of the function *f*. Let us denote the set of (ε, η)-almost periods of f by $E\{\varepsilon, \eta; f\}$.

A function *f* is called almost periodic in the Lebesgue measure μ $(\mu - a.p.)$ iff for ant two positive numbers ε, η the set $E\{\varepsilon, \eta; f\}$ is relatively dense. Let us denote the set of μ-*a.p.* functions by $\widetilde{\aleph}^\mu$.

Elementary properties of μ-*a.p.* functions can be found in [2]. In particular, if $f \in \widetilde{\aleph}^\mu$, then $f \in \widetilde{\aleph}$ and *f* is a μ-continuous function, i. e. for any two positive numbers ε, η there exists $\delta > 0$ such that for every $w \in R$, $|w| < \delta$, we have $D(\eta; f, f_\tau) \le \varepsilon$.

For a nonempty subset $\mathrm{M} \not\subset \widetilde{\aleph}^\mu$ and for $N > 0$ put

$$\mathrm{M}_N = \{f_N : f \in \mathrm{M}\},$$

where f_N is the truncated function of *f* of the form

$$f_N(t) = \begin{cases} N & \text{for } t \text{ such that } f(t) > N \\ f(t) & \text{for } t \text{ such that } |f(t)| \le N. \\ -N & \text{for } t \text{ such that } f(t) < -N \end{cases}$$

It is known (see [2]) that if $f \in \tilde{\aleph}^\mu$, then for every $N > 0$, $f_N \in \tilde{\aleph}^\mu$ and for every $\varepsilon > 0$ and every $\eta > 0$ there exists $N > 0$ such that $D(\eta; f, f_\tau) \le \varepsilon$.

2. CONDITIONALLY μ-COMPACTNESS OF THE FAMILY OF ALMOST PERIODIC FUNCTIONS IN THE LEBESGUE MEASURE

We say that a family $A \not\subset \widetilde{\aleph}$ is conditionally μ-compact iff the set A is conditionally compact with respect to the F-norm $|\ |$ of the form (1) in $\tilde{S}$, i.e. every sequence (f_n), where $f_n \in A$ for $n = 1,2,...$, contain a Cauchy subsequence.

Theorem 2. A nonempty subset $\mathcal{M} \not\subset \aleph^{\mu}$ is conditionally μ-compact, if and only if, the following two conditions are satisfied:

a) For every $N > 0$ the family of truncated functions $\mathcal{M}_N$ is conditionally μ-compact.

b) For any two positive numbers ε, η there exists $N > 0$ such that for every $f \in \mathcal{M}$ we have

$$D(\eta, f, f_N) \leq \varepsilon$$

Proof. Necessity. We assume that the family of μ-a.p. functions $\mathcal{M}$ is conditionally μ-compact. By the Hausdorff Theorem, there exists a finite $(\varepsilon/3)$-net for the family $\mathcal{M}$

$$f^1, f^2, ..., f^n,$$

i.e for every $f \in \mathcal{M}$ there exists $k \in \{1,2,...,n\}$ such that

$$|f - f^k| < \frac{\varepsilon}{3}. \tag{2}$$

By Theorem 1 and (2) it follows that for every $\eta > 0$ there exists $k \in \{1,2,...,n\}$ such that

$$D(\eta/3; f, f_k) \leq \frac{\varepsilon}{3}. \tag{3}$$

Using (3), for an arbitrary $N > 0$ we have for every $f_N \in \mathcal{M}_N$ and for any two positive numbers η, ε

$$D(\eta; f, f_N^k) \leq \frac{\varepsilon}{3}. \tag{4}$$

From Theorem 1 and (4) it follows that for every $\varepsilon > 0$ there exists $k \in \{1,2,...,n\}$ such that

$$|f_N - f_N^k| < \varepsilon,$$

and so the set

$$f_N^1, f_N^2, \ldots, f_N^n \in \mathcal{M}_N$$

is an ε-net for M_N. Hence the family $\mathcal{M}_N \not\subseteq \tilde{S}$ is conditionally compact with respect to the F-norm $|\ \ |$ in $\tilde{S}$. This shows a).

In the following we shall prove the condition b). For arbitrary $\varepsilon > 0$, $\eta > 0$ and every f^l, $l = 1,2,\ldots,n$, there exists $N_1 = N_1(\varepsilon, \eta) > 0$ such that

$$D(\eta/3; f^l, f^l_{N_1}) \leq \frac{\varepsilon}{3} \qquad \text{for} \qquad l = 1,2,\ldots,n.$$

Let us put

$$N = \max\{N_1 : l = 1,2,\ldots,n\}.$$

Then

(5) $$D(\eta/3; f^l, f_N^l) \leq \frac{\varepsilon}{3} \qquad \text{for} \qquad l = 1,2,\ldots,n.$$

From (3) and (5) it follows that for every f

$$D(\eta; f, f_N) \leq D(\eta/3; f, f^k) + D(\eta/3; f^k, f_N^k) +$$

$$+ D(\eta/3; f_N^k, f_N) \leq (2/3) + D(\eta/3; f^k, f) \leq \varepsilon.$$

Hence we obtain the condition b).

Sufficiency. By the condition b), for arbitrary $\varepsilon > 0$, $\eta > 0$ there exists $N = N(\varepsilon, \eta) > 0$ such that for every f we have

(6) $$D(\eta; f, f_N) \leq \frac{\varepsilon}{9}.$$

From the condition a) it follows that the set M_N is conditionally μ-compact. By Theorem 1 and (6) we obtain that for an arbitrary $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

(7) $$|f - f_N| < \frac{\varepsilon}{3}.$$

By the Hausdorff Theorem there exists a finite $(\varepsilon/3)$-net for the family $\mathcal{M}_N$

$$f_N^1, f_N^2, \ldots, f_N^n \in \mathcal{M}_N,$$

i.e for every $f_N \in \mathcal{M}_N$ there exists $l \in \{1,2,\ldots,n\}$ such that

(8) $$|f_N - f_N^l| < \frac{\varepsilon}{3}.$$

From (7) and (8) it follows that for every $f \in M$ we have

$$|f - f^1| \leq |f - f_N| + |f_N - f_N^1| + |f_N^1 - f^1| < \varepsilon$$

and so the set $\mathcal{M}$ is conditionally μ-compact.

3. μ-NORMAL FUNCTIONS

A function $f \in \widetilde{\aleph}$ is called μ-normal iff the family of functions

$$f_T = \{f_h : h \in R\},$$

where $f_h(t) = f(t+h)$ for every $t \in R$, is conditionally μ-compact.

Theorem 3. *It is necessary and sufficient that the function $f \in \widetilde{\aleph}$ be μ-a.p. that it is μ-normal.*

Proof. Necessity. Let f be μ-a.p.. Becouce f is μ-continuous, so for any two positive numbers ε, η there exists $\delta > 0$ such that for every $w \in R$, $|w| < \delta$, we have $D(\eta/2; f, f_w) \leq (\varepsilon/6)$. Let us put $h_k = k\delta$ for $k = 1,2,\dots,n$, where n we choose such that $n\delta \leq \omega < (n+1)\delta$, $\omega > 0$ is a number which characterizes the relative density of the set $E\{\varepsilon/6, \eta/2; f\}$. Because for every $f_h \in f_T$ and for any two positive numbers ε, η we have for $\tau \in E\{\varepsilon/6, \eta/2; f\}$ such that $-h < \tau < -h + \omega$ and for h_k, $k \in \{1,2,\dots,n\}$, such that $|h + \tau - h_k| < \delta$ the following estimation

$$D(\eta; f_h, f_{hk}) \leq D(\eta/2; f_h, f_{h\tau}) + D(\eta/2; f_{h\tau}, f_{h_k}) \leq \frac{\varepsilon}{3},$$

where $f_{h\tau}(t) \equiv f(t + h + \tau)$, so the set

$$f_{h_1}, f_{h_2}, \dots, f_{h_n} \in f_T$$

is an ε-net for the family $f_T \leq S$, and so the family f_T is conditionally μ-compact. We prove the sufficient condition in the same way as for S^p-a.p. functions (see [1], p. 220).

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(Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Poznań,Poland)

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