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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF FOURTH ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

ABSTRACT: The authors consider the nonlinear difference equation

(E) $$\Delta^2(a_n\Delta(b_n\Delta y_n)) + f(n, y_{n-\ell}) = 0, \quad n \in N(n_0) = \{n_0, n_0+1, \dots\},$$

where $\{a_n\}$ and $\{b_n\}$ are positive real sequences, ℓ is a nonnegative integer, $f: N(n_0)\times R \to R$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$. They obtain necessary and sufficient conditions for the existence of nonoscillatory solutions with a specified asymptotic behavior. They also obtain sufficient conditions for all solutions to be oscillatory if f is either strongly sublinear or strongly superlinear. Examples of their results are also included.

KEY WORDS: asymptotic behavior, delay, difference equations, existence of nonoscillatory solutions, fourth order, oscillatory solutions.

1. INTRODUCTION

Consider the difference equation

(1) $$\Delta^2(a_n\Delta(b_n\Delta y_n)) + f(n, y_{n-\ell}) = 0, \qquad n \in N(n_0),$$

where $N(n_0) = \{n, n_0+1, \dots\}$, n_0 is a nonnegative integer, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ and $\{b_n\}$ are sequences of positive real numbers, ℓ is a nonnegative integer, $f: N(n_0)\times R \to R$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$ and all $n \in N(n_0)$, and $f(n,\cdot) \not\equiv 0$ eventually.

By a solution of equation (1), we mean a real sequence $\{y_n\}$ defined for all $n \geq n_0 - \ell$ and satisfies equation (1) for all $n \in N(n_0)$. A solution of equation (1) is *nonoscillatory* if it is eventually positive or eventually negative, and is *oscillatory* otherwise. Determining oscillation criteria for difference equations has received a great deal of attention in the last few years (see the monographs by Agarwal [1], Agarwal and Wang [2], and Kocic and Ladas [4]).Compared to second order difference equations, the study of higher order equations, and in particular fourth order equations, has received considerably less attention (see, for example, Hooker and Patula [3], Popenda and Schmeidel [6], Smith and Taylor [7], Taylor [8], and the references contained therein). Our purpose here is to establish some necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (1) that exhibit certain types of asymptotic behavior. In addition, we obtain some sufficient conditions for equation (1) to be oscillatory. Results obtained here are motivated by some results obtained by Kusano and Naito in [5]. We illustrate our results with examples.

2. EXISTENCE OF NONOSCILLATORY SOLUTIONS

In this section, we obtain necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (1) with certain asymptotic properties. We begin with the following lemma.

LEMMA 2.1. *Assume that either*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty, \tag{2a}$$

$$\sum_{n=n_0}^{\infty} \frac{n}{a_n} = \infty \quad \text{and} \quad 0 < m \le b_n \le M, \tag{2b}$$

or

$$\sum_{n=n_0}^{\infty} \frac{P_n}{a_n} = \sum_{n=n_0}^{\infty} \frac{n}{a_n} = \infty \quad \text{where} \quad P_n = \sum_{s=n_0}^{n} \frac{1}{b_s} \to \infty \quad \text{as} \quad t \to \infty. \tag{2c}$$

If $\{y_n\}$ *is an eventually positive solution of equation* (1), *then exactly one of the following statements holds:*

(i) $\Delta y_n > 0$, $\Delta(b_n \Delta y_n) > 0$, *and* $\Delta(a_n \Delta(b_n \Delta y_n)) > 0$ *for all sufficiently large n*;

(ii) $\Delta y_n > 0$, $\Delta(b_n \Delta y_n) < 0$, *and* $\Delta(a_n \Delta(b_n \Delta y_n)) > 0$ *for all sufficiently large n.*

PROOF. Let $\{y_n\}$ be an eventually positive solution of equation (1). Then there exists an integer $n_1 \in N(n_0)$ such that $y_{n-\ell} > 0$ for all $n \in N(n_1)$. Now $\Delta^2(a_n \Delta(b_n \Delta y_n)) < 0$ for all $n \in N(n_1)$, so $\{\Delta(a_n \Delta(b_n \Delta y_n))\}$, $\{a_n \Delta(b_n \Delta y_n)\}$, and $\{b_n \Delta y_n\}$ are eventually monotonic and of one sign, say for $n \ge n_2$. Suppose that $\Delta(a_{n_3} \Delta(b_{n_3} \Delta y_{n_3})) = -c_1 \le 0$ for some $n_3 \ge n_2$. Note that since $f(n,\cdot) \not\equiv 0$, we can assume that $c_1 \ne 0$. It follows that $\Delta(a_n \Delta(b_n \Delta y_n)) \le -c_1$ for $n \in N(n_3)$. Summing both sides of the last inequality, we see that there exists $n_4 \in N(n_3)$ and $c_2 > 0$ such that $\Delta(b_n \Delta y_n) \le -(c_2 n)/a_n$ for $n \in N(n_4)$. Summing again, we obtain

$$b_n \Delta y_n \le b_{n_4} \Delta y_{n_4} - c_2 \sum_{s=n_4}^{n-1} \frac{s}{a_s}.$$

If either (2a), (2b), or (2c) holds, there exists $n_5 > n_4$ and $c_3 > 0$ such that $b_n \Delta y_n \le -c_3$ for $n \in N(n_5)$. A final summation yields

$$y_n \le y_{n_5} - c_3 \sum_{s=n_5}^{n-1} \frac{1}{b_s}$$

which, in view of (2), implies $\lim_{n\to\infty} y_n = -\infty$. This contradiction implies $\Delta(a_n \Delta(b_n \Delta y_n)) > 0$ for $n \in N(n_2)$.

Now if $a_n \Delta(b_n \Delta y_n) < 0$ for $n \in N(n_6)$ for some $n_6 \ge n_5$, then $\{b_n \Delta y_n\}$ must be eventually positive for otherwise we are again led to conclude that $\lim_{n\to\infty} y_n = -\infty$. Thus, Case (ii) is verified.

Next, assume that $a_n \Delta(b_n \Delta y_n) > 0$ for all $n \in N(n_3)$. Then we have

(3) $$a_n \Delta(b_n \Delta y_n) \ge a_{n_3} \Delta(b_{n_3} \Delta y_{n_3}) = c_4 > 0$$

for $n \in N(n_3)$. If (2a) holds, divide the above inequality by a_n and sum from n_3 to $n-1$ to obtain

$$b_n \Delta y_n - b_{n_3} \Delta y_{n_3} > c_4 \sum_{s=n_3}^{n-1} \frac{1}{a_s} \to \infty$$

as $n \to \infty$. Hence, $\{\Delta y_n\}$ is eventually positive.

If (2b) holds, we multiply instead by n/a_n and sum from n_3 to $n-1$ to obtain

$$n b_n \Delta y_n - n_3 b_{n_3} \Delta y_{n_3} - \sum_{s=n_3}^{n-1} b_{s+1} \Delta y_{s+1} > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s}.$$

If $\Delta y_n < 0$ for all $n \ge n_3$, then

$$n b_n \Delta y_n - n_3 b_{n_3} \Delta y_{n_3} - M(y_{n+1} - y_{n_3+1}) > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s},$$

so

$$n b_n \Delta y_n - n_3 b_{n_3} \Delta y_{n_3} + M y_{n_3+1} > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s} \to \infty$$

as $n \to \infty$. Thus, $\{y_n\}$ is eventually positive.

If (2c) holds, multiplying (3) by P_n/a_n and summing from n_3 to $n-1$, we have

$$P_n b_n \Delta y_n - P_{n_3} b_{n_3} \Delta y_{n_3} - \sum_{s=n_3}^{n-1} b_{s+1} \Delta y_{s+1} \Delta P_s > c_4 \sum_{s=n_3}^{n-1} \frac{P_s}{a_s}.$$

Since $\Delta P_n = 1/b_{n+1}$, we have

$$P_n b_n \Delta y_n - P_{n_3} b_{n_3} \Delta y_{n_3} + y_{n_3+1} > c_4 \sum_{s=n_3}^{n-1} \frac{s}{a_s} \to \infty$$

as $n \to \infty$. Therefore, $\{\Delta y_n\}$ is again eventually positive, and the proof of Case (i) is complete.

NOTE. In the remainder of the paper, when we say that condition (2) holds, we mean that at least one of the conditions (2a), (2b), or (2c) holds.

For what follows, we need to introduce the notation

$$Q_{n,N} = \sum_{s=N}^{n-2} \frac{s}{a_s}\left(\sum_{j=s+1}^{n-1} \frac{1}{b_j}\right) \quad \text{and} \quad R_{n,N} = \sum_{s=N+1}^{n-2} \frac{(s-N)}{a_s}\left(\sum_{j=s+1}^{n-1} \frac{1}{b_j}\right).$$

Note that $Q_{n,N}$ can be written as $Q_{n,N} = \sum_{s-N}^{n-1} 1/b_s \,(\sum_{t=N}^{s-1} t/a_t)$ which is quite useful in doing calculations.

LEMMA 2.2. *Assume condition* (2) *holds and*

(4) $$\Delta(a_{n-1}\Delta^2 R_{n-1,N}) = \frac{1}{b_n}, \qquad n \in N(n_0).$$

If $\{y_n\}$ *is an eventually positive solution of equation* (1), *then there exist positive constants* C_1 *and* C_2 *and an integer* $M \in N(n_0)$ *such that*

(5) $$C_1 \le y_n \le C_2 Q_{n,n_0}$$

and

(6) $$y_n \ge R_{n,M}\,\Delta(a_n \Delta(b_n \Delta y_n))$$

for $n \in N(M)$.

PROOF. Let $\{y_n\}$ be an eventually positive solution of equation (1). Then there exists an integer $n_1 \in N(n_0)$ such that $y_{n-\ell} > 0$ for $n \in N(n_1)$. Now from

Lemma 2.1, we have $b_n \Delta y_n > 0$ for $n \in N(N)$ for some $N \geq n_1$, and so $y_n \geq C_1 > 0$ for $n \in N(N)$. To prove the right side of (5), we sum $\Delta^2(a_n \Delta(b_n \Delta y_n)) < 0$ twice from N to $n-1$ to obtain

$$\Delta(b_n \Delta y_n) < \frac{A_0 n}{a_n} + \frac{A_1}{a_n}, \quad n \in N(N),$$

where A_0 and A_1 are constants. Summing the last inequality again from N to $n-1$, we have

$$\Delta y_n < \frac{A_0}{b_n} \sum_{s=N}^{n-1} \frac{s}{a_s} + \frac{A_1}{b_n} \sum_{s=N}^{n-1} \frac{1}{a_s} + \frac{A_2}{b_n}, \quad n \in N(N),$$

where A_2 is a constant. A final summation of the last inequality yields

$$y_n < A_0 \sum_{s=N}^{n-1} \frac{1}{b_s} \left(\sum_{t=N}^{s-1} \frac{t}{a_t} \right) + A_1 \sum_{s=N}^{n-1} \frac{1}{b_s} \left(\sum_{t=N}^{s-1} \frac{1}{a_t} \right) + A_2 \sum_{s=N}^{n-1} \frac{1}{b_s} + A_3$$

for some constant A_3. It is easy to see that there exists $C_2 > 0$ and a positive integer M such that $y_n \leq C_2 Q_{n,n_0}$ for $n \geq M$.

To prove (6), let $N \in N(n_0)$ be large enough so that $\{y_n\}$ satisfies Case (i) or Case (ii) of Lemma 2.1. Assume first that Case (i) holds. Since $\{\Delta(a_n \Delta(b_n \Delta y_n))\}$ is positive and nonincreasing, it follows that

$$a_n \Delta(b_n \Delta y_n) \geq a_n \Delta(b_n \Delta y_n) - a_N \Delta(b_N \Delta y_N) =$$

$$= \sum_{s=N}^{n-1} \Delta(a_s \Delta(b_s \Delta y_s)) \geq \Delta(a_{n-1} \Delta(b_{n-1} \Delta y_{n-1}))(n-N),$$

or

$$\Delta(b_n \Delta y_n) \geq \frac{(n-N)}{a_n} \Delta(a_n \Delta(b_n \Delta y_n)), \quad n \in N(N).$$

Summing the last inequality from $N+1$ to $n-1$, we obtain

$$b_n \Delta y_n \geq \sum_{s=N+1}^{s-1} \frac{s-N}{a_s} \Delta(a_s \Delta(b_s \Delta y_s)) \geq \Delta(a_n \Delta(b_n \Delta y_n)) \sum_{s=N+1}^{n-1} \frac{s-N}{a_s},$$

or

$$y_n \geq \sum_{s=N}^{n-1} \frac{\Delta(a_s\Delta(b_s\Delta y_s))}{b_s} \sum_{t=N+1}^{s-1} \frac{t-N}{a_t} \geq \Delta(a_n\Delta(b_n\Delta y_n)) \sum_{s=N+1}^{n-1} \frac{1}{b_s}\left(\sum_{t=N}^{s-1} \frac{t-N}{a_t}\right).$$

which proves (6) if (i) holds.

Next, assume that Case (ii) holds. Multiplying (1) by $R_{n+2,N}$ and summing from N to $n-1$, we obtain

$$\sum_{s=N}^{n-1} R_{s+2,N}\Delta^2(a_s\Delta(b_s\Delta y_s)) + \sum_{s=N}^{n-1} R_{s+2,N} f(s, y_{s-\ell}) = 0.$$

By a repeated summation by parts and using condition (4), we obtain

$$R_{n+1,N}\Delta(a_n\Delta(b_n\Delta y_n)) - \Delta R_{n,N} a_n \Delta(b_n\Delta y_n) +$$
$$+ a_{n-1}b_n(\Delta^2 R_{n-1,N})\Delta y_n - y_n + \sum_{s=N}^{n-1} R_{s+2,N} f(s, y_{s-\ell}) \leq 0.$$

Lemma 2.1 (ii) implies

$$y_n \geq R_{n+1,N}\Delta(a_n\Delta(b_n\Delta y_n)) \geq R_{n,N}\Delta(a_n\Delta(b_n\Delta y_n)), \quad n \in N(N).$$

This completes the proof of Lemma 2.2.

REMARK. Note that if $b_n \equiv 1$, then condition (4) is automatically satisfied.

THEOREM 2.3. *Assume that f is increasing and condition* (2) *holds. Then a necessary and sufficient condition for equation* (1) *to have a nonoscillatory solution $\{y_n\}$ satisfying $\lim_{n\to\infty} \frac{y_n}{Q_{n,n_0}} = d \neq 0$ is that*

(7) $$\sum_{n=n_0}^{\infty} |f(n, CQ_{n-\ell,n_0})| < \infty$$

for some $C \neq 0$.

PROOF. ***Necessity.*** Let $\{y_n\}$ be a nonoscillatory solution of equation (1) with $\lim_{n\to\infty} \frac{y_n}{Q_{n,n_0}} = d \neq 0$. Without loss of generality, we may assume that $d > 0$. Then there exist positive numbers d_1, d_2 and $n_1 \in N(n_0)$ such that $d_1 Q_{n-\ell,n_0} \leq y_{n-\ell} \leq d_2 Q_{n-\ell,n_0}$ for $n \in N(n_1)$. Thus

(8) $$f(n, y_{n-\ell}) \geq f(n, d_1 Q_{n-\ell,n_0}), \quad n \in N(n_1).$$

On the other hand, summing equation (1) from n_1 to $n-1$ and using the fact that $\Delta(a_n\Delta(b_n\Delta y_n))>0$, we have $\sum_{s=n_1}^{n-1} f(s,y_{n-\ell})<\infty$. Then, from (8), we conclude that $\sum_{s=n_1}^{\infty} f(n,d_1Q_{n-\ell,n_0})<\infty$.

Sufficiency. Assume that (7) holds with $C>0$ since a similar argument holds if $C<0$. Take $N\in N(n_0)$ sufficiently large so that

$$\sum_{n=N}^{\infty} f(n,CQ_{n-\ell,n_0})<\frac{C}{8}. \tag{9}$$

Consider the Banach space B_N of all real sequences $Y=\{y_n\}$, $n\in N(N)$ with sup-norm $\|Y\|=\sup_{n\geq N}\{|y_n|/Q_{n,n_0}^2\}$. Define

$$S=\{Y\in B_N:(C/2)Q_{n,n_0}\leq CQ_{n,n_0},\ \ n\in N(N)\}.$$

Clearly, S is a bounded, closed, and convex subset of B_N. Now, we define a partial ordering on B_n in the usual sense, that is, $X\leq Y$ means $x_n\leq y_n$ for all $n\geq N$. Then, for every subset A of S, both $\sup S$ an $\inf S$ exist in S. We also define operator $T:S\to B_N$ by

$$(Ty)_n=\begin{cases}\frac{C}{2}Q_{n,n_0}+Q_{n,n_0}\sum\limits_{s=n-2}^{\infty} f(s,y_{s-\ell})+\sum\limits_{s=N}^{n-3} Q_{s+2,n_0}f(s,y_{s-\ell})+\\ \quad+\sum\limits_{s=N}^{n-3}\left(\sum\limits_{i=N}^{s}\frac{i}{a_i}\left[\sum\limits_{j=s}^{n-3}\frac{1}{b_{j+2}}f(s,y_{s-\ell})\right]\right)+\\ \quad+\sum\limits_{s=N}^{n-3}\left(\sum\limits_{i=s}^{n-3}\frac{i}{a_{i+1}}\left[\sum\limits_{j=i}^{n-4}\frac{1}{b_{j+3}}(s+1)f(s,y_{s-\ell})\right]\right), & N\leq n,\\ (Ty)_N, \quad N-\ell\leq n<N. & \end{cases} \tag{10}$$

The mapping T satisfies the assumptions of the Knaster-Tarski fixed point theorem. That is:

(i) $TS\subset S$. If $Y\in S$, then $Ty_n\geq(C/2)Q_{n,n_0}$ for $n\geq N$. Furthermore,

$$\sum_{s=N}^{n-3}\left(\sum_{i=s}^{s}\frac{i}{a_i}\left[\sum_{j=s}^{n-3}\frac{1}{b_{j+2}}f(s,y_{s-\ell})\right]\right)\leq \tag{11}$$

$$\le \sum_{s=N}^{n-3}\left(\sum_{i=N}^{n-2}\frac{1}{a_i}\left[\sum_{j=i+1}^{n-1}\frac{1}{b_j}f(s,y_{s-\ell})\right]\right) \le Q_{n,n_0}\sum_{s=N}^{n-3}f(s,y_{s-\ell}).$$

and

$$\sum_{s=N}^{n-3}\left(\sum_{i=s}^{n-3}\frac{1}{a_{i+1}}\left[\sum_{j=i}^{n-4}\frac{1}{b_{j+3}}(s+1)f(s,y_{s-\ell})\right]\right) \le \tag{12}$$

$$\le \sum_{s=N}^{n-3}\left(\sum_{i=N}^{n-2}\frac{1}{a_i}\left[\sum_{j=i+1}^{n-1}\frac{1}{b_j}f(s,y_{s-\ell})\right]\right) \le Q_{n,n_0}\sum_{s=N}^{n-3}f(s,y_{s-\ell}).$$

From (9) – (12), it follows that

$$(Ty)_n \le \tfrac{1}{2}CQ_{n,n_0} + \tfrac{1}{8}CQ_{n,n_0} + \tfrac{1}{8}CQ_{n,n_0} + \tfrac{1}{8}CQ_{n,n_0} + \tfrac{1}{8}CQ_{n,n_0} = CQ_{n,n_0}.$$

(ii) Clearly, T is increasing.

Therefore, by the Knaster-Tarski fixed point theorem, we conclude that there exist $Y \in S$ such that $TY = Y$. That is, $\{y_n\}$ is a solution of equation (1). Furthermore, from $Q_{n+1,n_0} > Q_{n,n_0}$, $\lim_{n\to\infty} Q_{n,n_0} = \infty$, and using Stoltz theorem, we have

$$\lim_{n\to\infty}\frac{y_n}{Q_{n,n_0}} = \lim_{n\to\infty}\frac{\Delta y_n}{\Delta Q_{n,n_0}} = \lim_{n\to\infty}\frac{b_n\Delta y_n}{b_n\Delta Q_{n,n_0}} = \lim_{n\to\infty}\frac{\Delta(b_n\Delta y_n)}{\Delta(b_n\Delta Q_{n,n_0})} = \lim_{n\to\infty}\frac{a_n\Delta(b_n\Delta y_n)}{n}.$$

However, from (10), it is easy to see that $\lim_{n\to\infty} a_n\Delta(b_n\Delta y_n)/n = C/2$. This shows that $\{y_n\}$ is a solution of equation (1) with the desired asymptotic behavior, and completes the proof of the theorem.

EXAMPLE 1. *Consider the difference equation*

$$\Delta^2(n\Delta((n+1)\Delta y_n) + \frac{2(n-1)^3}{n^3(n-2)^3(n+1)(n+2)(n+3)}y_{n-1}^3 = 0, \quad n \ge 3. \tag{E_1}$$

All conditions of Theorem 2.3 are satisfied with $Q_{n,3} = \sum_{s=3}^{n-1}(s-3)/(s+1)$ *(condition* (2a) *is satisfied here), and hence equation* (E_1) *has a solution* $\{y_n\}$ *such that* $\lim_{n\to\infty} y_n/Q_{n,3} = d \ne 0$. *In fact,* $\{y_n\} = \{(n^2-1)/n\}$ *is a solution of equation* (E_1) *with* $\lim_{n\to\infty} y_n/Q_{n,3} = 1$.

In the following theorem, we give a necessary and sufficient condition for the behavior of nonoscillatory solutions of equation (1) when $b_n \equiv 1$.

THEOREM 2.4. *Let* $b_n \equiv 1$, f *be increasing, and condition* (2) *hold. A necessary and sufficient condition for equation* (1) *to have a nonoscillatory solution* $\{y_n\}$ *such that* $\lim_{n\to\infty} y_n = d \neq 0$ *is that*

(13)
$$\sum_{n=n_0}^{\infty} Q_{n,n_0} |f(n,C)| < \infty$$

for some $C \neq 0$.

PROOF. ***Necessity***. Let $\{y_n\}$ be a nonoscillatory solution of equation (1) with $\lim_{n\to\infty} y_n = d > 0$. Then, there are constants d_1, d_2, and an integer $n_1 \in N(n_0)$ such that $d_1 \leq y_{n-\ell} \leq d_2$ for $n \in N(n_1)$. Hence, we have $f(n, y_{n-\ell}) \geq f(n, d_1)$. We now multiply equation (1) by Q_{n+2,n_1} and then sum from n_1 to $n-1$. Using a method similar to Lemma 2.2, we obtain

(14)
$$\sum_{s=n_1}^{n-1} Q_{s+2,n_1} f(s, y_{s-\ell}) = -Q_{n+1,n_1} \Delta(a_n \Delta^2 y_n) +$$
$$+ \Delta Q_{n,n_1} a_n \Delta^2 y_n - a_{n-1}(\Delta^2 Q_{n-1,n_1}) \Delta y_n + y_n - y_{n_1}.$$

Observing that $\{y_n\}$ belongs to Case (ii) of Lemma 2.1, from (14) we obtain

$$\sum_{s=n_1}^{\infty} R_{s,n_0} f(s, y_{s-\ell}) \leq \sum_{s=n_1}^{\infty} R_{s+2,n_0} f(s, y_{s-\ell}) < \infty,$$

and hence the result foolows.

Sufficiency. Suppose that (13) holds with $C > 0$. Take $N \in N(n_0)$ sufficiently large so that

$$\sum_{n=N}^{\infty} Q_{n,N} f(n,C) < \frac{C}{8}.$$

Let B_N be the Banach space considered in the proof of Theorem 2.3 with norm $\|Y\| = \sup_{n\geq N} |y_n|$. We define a bounded, closed, and convex subset S of B_N by $S = \{y \in B_N : (C/2) \leq y_n \leq C, n \in N(N)\}$, and an operator $T : S \to B_N$ by

$$(Ty)_n = \begin{cases} \dfrac{C}{2} + Q_{n-1,n_0} \displaystyle\sum_{j=n-3}^{\infty} f(j, y_{j-\ell}) + \sum_{j=N}^{n-4} Q_{j+2,n_0} f(j, y_{j-\ell}) + \\ \quad + (n-1) \displaystyle\sum_{j=n-2}^{\infty} \left(\sum_{i=n-3}^{j-1} \frac{j-i-1}{a_{i+2}} \right) f(j, y_{j-\ell}) + \\ \quad + \left(\displaystyle\sum_{j=N}^{n-2} \frac{j}{a_j} \right) \left(\sum_{j=n-2}^{\infty} (j-n+3) f(j, y_{j-\ell}) \right), & N \le n, \\ (Ty)_N, \quad N - \ell \le n < N. \end{cases}$$

Arguing as in Theorem 2.3, we can easily show that the mapping T satisfies the conditions of the Knaster-Tarski fixed point theorem. Therefore, there exists $Y \in S$ such that $TY = Y$, that is, $\{y_n\}$ is a nonoscillatory solution of equation (1). Since

$$\Delta y_n = \sum_{j=n-1}^{\infty} \left(\sum_{i=n-2}^{j-1} \frac{(j-i-1)}{a_{i+2}} \right) f(j, y_{j-\ell}) > 0, \qquad n \ge N(N),$$

we have $\lim_{n\to\infty} y_n = d \in [C/2, C]$. This completes the proof of Theorem 2.4.

EXAMPLE 2. *Consider the difference equation*

$$(E_2) \qquad \Delta^2(n\Delta^2 y_n) + \frac{12n^{\frac{1}{3}}}{(n-1)^{\frac{1}{3}}(n+1)(n+2)(n+3)(n+4)} y_n^{\frac{1}{3}} = 0, \quad n \ge 2.$$

All conditions of Theorem 2.4 are satisfied with $Q_{n,2} = \frac{(n-2)(n-3)}{2}$ *and hence the equation* (E_2) *has solutions* $\{y_n\}$ *such that* $\lim_{n\to\infty} y_n = d \ne 0$. *Hence,* $\{y_n\} = \{\frac{n-1}{n}\}$ *is such a solution of* (E_2).

3. OSCILLATION THEOREMS

In this section, we establish some oscillation results for equation (1) when f is strongly sublinear or strongly superlinear as given in the following definition.

DEFINITION 3.1. *The function f is called* ***strongly superlinear*** *if there exists a constant* $\lambda > 1$ *such that for* $u \ge v > 0$ *or* $u \le v < 0$,

$$\frac{f(n,u)}{|u|^{\lambda} \operatorname{sgn} u} \ge \frac{f(n,v)}{|v|^{\lambda} \operatorname{sgn} v}, \qquad n \in N(n_0).$$

The function f is called **strongly sublinear** *if there exists a constant μ with $0<\mu<1$ such that for $u \geq v > 0$ or $u \leq v < 0$,*

$$\frac{f(n,u)}{|u|^{\mu}\operatorname{sgn} u} \geq \frac{f(n,v)}{|v|^{\mu}\operatorname{sgn} v}, \qquad n \in N(n_0).$$

THEOREM 3.2. *Assume that conditions* (2) *and* (4) *hold. Let f be strongly sublinear and*

$$\sum_{n=n_0}^{\infty} |f(n, CQ_{n-\ell,n_0})| = \infty \tag{15}$$

for all $C \neq 0$. Then all solutions of equation (1) *are oscillatory.*

PROOF. Assume that $\{y_n\}$ is a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y_{n-\ell} > 0$ for all $n \in N(n_1)$, $n_1 \geq n_0$. By Lemma 2.1, $\{\Delta(a_n \Delta(b_n \Delta y_n))\}$ is positive and decreasing. Summing equation (1), we obtain

$$\Delta(a_n \Delta(b_n \Delta y_n)) \geq \sum_{s=n}^{\infty} f(s, y_{s-\ell}), \quad n \in N(n_1).$$

From (6), we have $y_n \geq R_{n,N} \Delta(a_n \Delta(b_n \Delta y_n))$ for $n \in N(N)$, $N \geq n_1$. Combining the last two inequalities, we obtain

$$y_n \geq R_{n,N} \sum_{s=n}^{\infty} f(s, y_{s-\ell}), \quad n \geq N. \tag{16}$$

In view of (5), there exists a constant $K > 0$ such that $y_n \leq KQ_{n-\ell,n_0}$ for $n \in N(N_1)$ for some $N_1 \geq N$. Let $s_n = \sum_{j=n}^{\infty} f(j, y_{j-\ell})$; then by the mean value theorem, we have

$$\Delta(s_n^{1-\mu}) = (1-\mu) t^{-\mu} \Delta s_n, \quad s_{n+1} < t < s_n,$$

where μ is the sublinearity constant. Using (16), the estimate $y_{n-\ell} \leq KQ_{n-\ell,n_0}$, and the strong sublinearity of f, we obtain that

$$
\begin{aligned}
-\Delta(s_n^{1-\mu}) &= (1-\mu)t^{-\mu}(-\Delta s_n) \ge (1-\mu)s_n^{-\mu} f(n, y_{n-\ell}) = \\
&= (1-\mu)s_n^{-\mu} y_{n-\ell}^{-\mu} f(n, y_{n-\ell}) \ge \\
&\ge (1-\mu)s_n^{-\mu}(R_{n-\ell,N} s_n)^{\mu}[KQ_{n-\ell,n_0}]^{-\mu} f(n, KQ_{n-\ell,n_0}) = \qquad (17)\\
&= (1-\mu)K^{-\mu}\left(\frac{R_{n-\ell,N}}{Q_{n-\ell,n_0}}\right)^{\mu} f(n, KQ_{n-\ell,n_0}).
\end{aligned}
$$

Since $\lim\limits_{n\to\infty} \frac{R_{n-\ell,N}}{Q_{n-\ell,n_0}} = \lim\limits_{n\to\infty} \frac{\Delta R_{n-\ell,N}}{\Delta Q_{n-\ell,n_0}} = 1$, there exists a constant $M_1 > 0$ and an integer $N_2 \in N(N_1)$ such that $(\frac{R_{n-\ell,N}}{Q_{n-\ell,n_0}})^{\mu} \ge M_1$ for $n \ge N_2$. Thus, from (17) we obtain

$$
-\Delta(s_n^{1-\mu}) \ge (1-\mu)K^{-\mu}M_1 f(n, KQ_{n-1,n_0}). \qquad (18)
$$

Summing (18) from N_2 to $n-1$, we have

$$
(1-\mu)M_1 K^{-\mu} \sum_{s=N_2}^{n-1} f(s, KQ_{s-\ell,n_0}) \le s_{N_2}^{1-\mu} - s_n^{1-\mu}.
$$

Hence, $\sum\limits_{s=N_2}^{\infty} f(s, KQ_{s-\ell,n_0}) < \infty$, which condradicts (15). This completes the proof of the theorem.

EXAMPLE 3. *Consider the difference equation*

$$
\Delta^2(n\Delta^2 y_n) + 16(n+1)y_{n-1}^{\frac{1}{3}} = 0, \quad n \ge 1. \qquad (E_3)
$$

With $Q_{n,1} = \frac{(n-1)(n-2)}{2}$, *all conditions of Theorem 3.2 are satisfied. Hence, all solutions are oscillatory. In fact,* $\{y_n\} = \{(-1)^n\}$ *is such a solution of equation* (E_3).

From Theorems 2.3 and 3.2 we obtain the following corollary.

COROLLARY 3.3. *Assume that conditions* (2) *and* (4) *hold f is increasing and strongly sublinear. Then all solution of equation* (1) *are oscillatory if and only if* (15) *is satisfied.*

THEOREM 3.4. *Assume that conditions* (2) *and* (4) *hold. If f is strongly superlinear and*

$$(19) \qquad \sum_{n=n_0}^{\infty} Q_{n-\ell,n_0} \, |f(n,C)| = \infty$$

for all $C \neq 0$, *then all solutions of equation* (1) *are oscillatory.*

PROOF. The proof is similar to that of Theorem 3.2 and therefore the details are omitted.

EXAMPLE 4. *Consider the difference equation*

$$(E_4) \qquad \Delta^2(n\Delta^2 y_n) + \frac{8(2n^2+6n+5)}{(n-3)^3} y_{n-3}^3 = 0, \qquad n \geq 4.$$

With $Q_{n,4} = \frac{(n-5)(n-4)}{2}$, *all conditions of Theorem 3.4 are satisfied and hence all solutions of equation* (E_4) *are oscillatory. In fact,* $\{y_n\} = \{(-1)^n n\}$ *is such a solution of equation* (E_4).

From Theorems 2.4 and 3.4 we have the following corollary.

COROLLARY 3.5. *Let* $b_n \equiv 1$ *and assume that condition* (2) *holds and f is strongly superlinear. Then all solutions of equation* (1) *are oscillatory if and only if* (19) *is satisfied.*

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