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RATE OF CONVERGENCE ON BASKAKOV-SZASZ TYPE OPERATORS

ABSTRACT: In the present article, we obtain a direct in simultaneous approximation for the linear combination of Baskakov-Szasz type operators in terms of higher order modulus of continuity

KEY WORDS: Stelov mean, modulus of continuity, rate of convergence, linear combination, Baskakov-Szasz type operators.

1. INTRODUCTION AND PRELIMINARIES

Gupta and Srivastava [4], introduced Baskokov-Szasz type operators defined for functions integrable on $[0,\infty)$ as

$$S_n(f,x) = n\sum_{\nu=0}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu}(t)f(t)dt, \quad x\in[0,\infty) \tag{1.1}$$

where $p_{n,\nu}(x)=\binom{n+\nu-1}{\nu}x^{\nu}(1+x)^{-n-\nu}$ and $q_{n,\nu}(t)=\dfrac{e^{-nt}(nt)^{\nu}}{\nu!}$.

Let $C_{\gamma}[0,\infty)=\{f\in C[0,\infty):|f(t)|\le Me^{\gamma t}$, for some $\gamma>0,\ M>0\}$. We define the norm $\|\cdot\|_{\gamma}$ on $C_{\gamma}[0,\infty)$ by $\|f\|_{\gamma}=\sup_{0\le t<\infty}|f(t)|e^{-\gamma t}$.

It turns out that the order of approximation by the operators (1.1) is $O(n^{-1})$ for smooth functions. Thus to improve the order of approximation, we consider the linear combination of the operators (1.1) as described below.

For $d_0,d_1,\ldots,d_k$ arbitrary but fixed distinct positive integers the linear combination $S_n(f,k,x)$ of $S_{d_jn}(f,x)$, $j=0(1)k$ are defined as

$$S_n(f,k,x)=\sum_{j=0}^{k}C(j,k)S_{d_jn}(f,x) \tag{1.2}$$

where $C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\dfrac{d_j}{d_j-d_i}$, $k\neq 0$ and $C(0,0)=1$.

LEMMA 2.1 [3]. *For $m \in N^0$ (the set of non-negative integers), if*

$$U_{n,m}(x)=\sum_{v=0}^{\infty} p_{n,v}(x)(vn^{-1}-x)^m$$

then there holds the recurrence relation

$$nU_{n,m+1}(x)=x(1+x)\left[U_{n,m}^{(1)}(x)+mU_{n,m-1}(x)\right], \quad for \quad m\geq 2.$$

Consequently,

(i) $U_{n,m}(x)$ is a polynomial in x of degree $\leq m$

(ii) for every fixed $x\geq 0$, $U_{n,m}(x)=O(n^{-[(m+1)/2]})$ where $[\alpha]$ denotes the integral part of α.

LEMMA 2.2. [4]. *For $m \in N^0$ (the set of non-negative integers), $n \in N$ and $x \in [0,\infty)$, if we define*

$$T_{n,m}(x) = n\sum_{v=0}^{\infty} p_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^m\,dt \tag{2.1}$$

then there holds

$$nT_{n,m+1}(x)=x(1+x)T_{n,m}^{(1)}(x)+(m+1)T_{n,m}(x)+mx(2+x)T_{n,m-1}(x), \quad m\geq 2.$$

Consequently we have

$$T_{n,0}(x)=1, \quad T_{n,1}(x)=n^{-1} \quad and \quad T_{n,2}(x)=\frac{2+nx(2+x)}{n^2}.$$

Also for every fixed $x\geq 0$, $T_{n,m}(x)=O(n^{-[(m+1)/2]})$.

LEMMA 2.3 [6]. *There exist the polynomials $\phi_{i,j,r}(x)$ independent of n and v such that*

$$\frac{d^r}{dx^r}\left[x^v(1+x)^{-n-v}\right]=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(v-nx)^j\phi_{i,j,r}(x)x^{v-r}(1+x)^{-n-v-r}.$$

LEMMA 2.4. *If f is r times $(r=1,2,\ldots)$ differentiable on $[0,\infty)$ $f^{(r)}$ is locally integrable in Lebesgue sense on $[0,\infty)$ and $f^{(r)}(t)=O(e^{\gamma t})$ for some $\gamma>0$ as $t\to\infty$. Then for $r=1,2,\ldots$ and $n>\gamma+r$, we have*

$$S_n^{(r)}(f,x) = \frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{v=0}^{\infty} p_{n+r,v}(x)\int_0^{\infty} q_{n,v+r}(t)f^{(r)}(t)dt.$$

PROOF. By (1.1), we have

$$S_n^{(r)}(f,x) = n\sum_{\nu=0}^{\infty} p_{n,\nu}^{(r)}(x)\int_0^{\infty} q_{n,\nu}(t)f(t)\,dt.$$

Applying Leibniz theorem, we have

$$S_n^{(r)}(f,x) =$$

$$= n\sum_{i=0}^{r}\sum_{\nu=i}^{\infty}\binom{r}{i}\frac{(n+\nu+r-i-1)!}{(n-1)!\,(\nu-i)!}(-1)^{r-i}x^{\nu-i}(1+x)^{-n-\nu-r-i}\int_0^{\infty} q_{n,\nu}(t)f(t)dt =$$

$$= n\sum_{\nu=0}^{\infty}\frac{(n+\nu+r-1)!}{(n-1)!\,\nu!}\frac{x^{\nu}}{(1+x)^{n+\nu+r}}\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}q_{n,\nu+i}(t)f(t)dt =$$

$$= n\frac{(n+r-1)!}{(n-1)!}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}q_{n,\nu+i}(t)f(t)dt.$$

Next, using again Leibniz theorem, we get

$$q_{n,\nu+r}^{(r)}(t) = \sum_{i=0}^{r}\binom{r}{i}(-1)^{i}n^{r}\frac{e^{-nt}(nt)^{\nu+i}}{(\nu+i)!} = n^{r}\sum_{i=0}^{r}(-1)^{i}\binom{r}{i}q_{n,\nu+i}(t).$$

Therefore

$$S_n^{(r)}(f,x) = \frac{(n+r-1)!}{(n-1)!\,n^{r-1}}\sum_{\nu=0}^{\infty} p_{n+r,\nu}(x)\int_0^{\infty} q_{n,\nu+r}^{(r)}(t)(-1)^{r}f(t)\,dt$$

integrating by part r times, we get the desired result.

Let $0<a<a_1<b_1<b<\infty$, then for sufficiently small $\delta>0$, the $(2k+2)$-th order Steklov mean $g_{2k+2,\delta}(t)$ corresponding to $g\in C_{\gamma}[0,\infty)$ is defined by

$$g_{2k+2,\delta}(t) = \delta^{-(2k+2)}\left(\int_{-\delta/2}^{\delta/2}\right)^{2k+2}\left[g(t)-\Delta_{\eta}^{2k+2}g(t)\right]\prod_{i=1}^{2k+2}dt_i$$

where $\eta=\sum_{i=1}^{2k+2}t_i$, $t\in[a,b]$ and $\Delta_{\eta}^{2k+2}g(t)$ is the $(2k+2)$-th forward difference of g with step length η. Then, we have

(i) $g_{2k+2,\delta}$ has continuous derivatives upto order $(2k+2)$ on $[a,b]$,

(ii) $\| g_{2k+2,\delta}^{(r)}\|_{C[a_1,b_1]} \le S_1\delta^{-1}\omega_r(g,\delta,a,b)$,

(iii) $\| g - g_{2k+2,\delta} \|_{C[a_1,b_1]} \leq S_2 \omega_{2k+2}(g,\delta,a,b)$,

(iv) $\| g_{2k+2,\delta} \|_{C[a_1,b_1]} \leq S_3 \| g \|_\gamma$

where S_i's are certain constants, independent of g and δ.

The above properties (i) – (iv) are extensions of the properties given in Freud and Popov [1] and Timan [7]. However for conciseness the proof is given as follows.

By repeated application of Theorem 18.17 of [5], it follows that $g_{2k+2,\delta}$ has continuous derivatives up to order $(2k+2)$ on $[a,b]$. Hence (i) is obtained. To prove (ii), writing

$$h_m(t) = \left(\int_{-\delta/2}^{\delta/2} \right)^{2k+2} g\left(t + m \sum_{i=1}^{2k+2} t_i \right) dt_1\, dt_2 \ldots dt_{2k+2}, \quad 1 \leq m \leq 2k+2.$$

Now using Theorem 18.17 of [5], we have

$$h_m^{(1)}(t) = m^{-1} \left(\int_{-\delta/2}^{\delta/2} \right)^{2k+1} \Delta_{m\delta}^{1}\, g\left(t - \frac{m\delta}{2} + m \sum_{i=0}^{2k+1} t_i \right) dt_1\, dt_2 \ldots dt_{2k+1}.$$

A repeated differentiation of above expression gives

$$(1.3) \quad h_m^{(r)}(t) = m^{-r} \left(\int_{-\delta/2}^{\delta/2} \right)^{2k+2-r} \Delta_{m\delta}^{r}\, g\left(t - \frac{rm\delta}{2} + m \sum_{i=0}^{2k+2-r} t_i \right) dt_1 dt_2 \ldots dt_{2k+2-r},$$

$$1 \leq r \leq 2k+1$$

and

$$(1.4) \qquad h_m^{(2k+2)}(t) = m^{-(2k+2)} \Delta_{m\delta}^{2k+2}\, g\left(t - \frac{m(2k+2)\delta}{2} \right) \quad \text{a.e.}$$

Now, from the definition of $g_{2k+2,\delta}(t)$ and $\delta_\eta^{2k+2} g(t)$

$$g_{2k+2,\delta}(t) = \frac{(-1)^{2k+1}}{\delta^{2k+2}} \left\{ \sum_{m=1}^{2k+2} \binom{2k+2}{m} (-1)^{2k+2-m} h_m(t) \right\}.$$

From this one obtains

$$(1.5) \qquad g_{2k+2,\delta}^{(r)}(t) = \frac{(-1)^{2k+1}}{\delta^{2k+2}} \left\{ \sum_{m=1}^{2k+2} \binom{2k+2}{m} (-1)^{2k+2-m} h_m^{(r)}(t) \right\}.$$

Using (1.3) – (1.5), we obtain

$$\left\|g^{(r)}_{2k+2,\delta}\right\|_{C[a_1,b_1]} \le S_1\delta^{-r}\omega_r(g,\delta,a,b).$$

Next, we prove (iii), by the definition of $g_{2k+2,\delta}(t)$, we have

$$\left|g_{2k+2,\delta}(t)-g(t)\right| \le \frac{1}{\delta^{2k+2}}\left(\int_{-\delta/2}^{\delta/2}\right)^{2k+2}\left|\Delta_\eta^{2k+2}g(t)\right|\prod_{i=1}^{2k+2}dt_i \le$$

$$\le S_2'\omega_{2k+2}(g,\delta(k+1),a,b) \le S_2\omega_{2k+2}(g,\delta,a,b),\quad t\in[a_1,b_1]$$

and thus (iii) is immediate.

Finally, it is trivial from the definition of $g_{2k+2,\delta}(t)$ that

$$\| g_{2k+2,\delta}\|_{C[a_1,b_1]} \le S_3\|g\|_{C[a,b]} \le S_3\|g\|_\gamma .$$

In [4] the authors have obtained a Voronovskaja type asymptotic formula and an estimate of error in simultaneous approximation. Recently Gupta and Srivastava [3] obtained similar direct results for the combinations (1.2). In the present paper, we extend the results of [3] and [4] and we obtain an estimate of error in terms of higher order modulus of continuity. The proof is carried out by using properties of a linear method of approximation viz. Steklov means.

2. MAIN RESULT

Our main theorem is as follows:

THEOREM 3.1. *Let $f^{(r)}\in C_\gamma[0,\infty)$ and $0<a<a_1<b_1<b<\infty$. Then for n sufficiently large*

$$\left\|S_n^{(r)}(f,k,\cdot)-f^{(r)}\right\|_{C[a_1,b_1]} \le \max\{C_1\omega_{2k+2}(f^{(r)},n^{-1/2},a,b),C_2n^{-(k+1)}\|f\|_\gamma\}$$

where $C_1=C_1(k,r)$ and $C_2=C_2(k,r,f)$.

PROOF. First

$$\left\|S_n^{(r)}(f,k,\cdot)-f^{(r)}\right\|_{C[a_1,b_1]} \le \left\|S_n^{(r)}(f-f_{2k+2,\delta},k,\cdot)\right\|_{C[a_1,b_1]} +$$

$$+\left\|S_n^{(r)}(f_{2k+2,\delta},k,\cdot)-f_{2k+2,\delta}^{(r)}\right\|_{C[a_1,b_1]}+\left\|f^{(r)}-f_{2k+2,\delta}^{(r)}\right\|_{C[a_1,b_1]}=$$

$$=J_1+J_2+J_3,\quad \text{say.}$$

It is obvious from the definition of Steklov mean that $f_{2k+2,\delta}^{(r)}(t)=$ $=(f^{(r)})_{2k+2,\delta}(t)$. Now by property (iii) of Steklov mean, we get

$$J_3\le C_1\,\omega_{2k+2}(f^{(r)},\delta,a,b).$$

Next by [3, Th. 3.1], we have

$$J_2\le C_2\,n^{-(k+1)}\sum_{j=r}^{2k+r+2}\left\|f_{2k+2,\delta}^{(j)}\right\|_{C[a,b]}.$$

Applying the interpolation property due to Goldberge and Meir [2], for each $j=r,r+1,\ldots(2k+r+2)$ followed by properties (ii) and (iv) of Steklov mean, we get

$$J_2\le C_3 n^{-(k+1)}\{\|f\|_\gamma+\delta^{-(2k+2)}\omega_{2k+2}(f^{(r)},\delta,a,b)\}.$$

Finally, we estimate J_1, choosing a^*, b^* satisfying $0<a<a^*<a_1<$ $<b_1<b^*<b<\infty$. Also let $\phi(t)$ be the characteristic function of internal $[a^*,b^*]$. Then

$$J_1\le\left\|S_n^{(r)}(\phi(t)(f(t))-f_{2k+2,\delta}(t),k,\cdot)\right\|_{C[a_1,b_1]}+$$

$$+\left\|S_n^{(r)}((1-\phi(t)(f(t)-f_{2k+2,\delta}(t))),k,\cdot)\right\|_{C[a_1,b_1]}=J_4+J_5,\quad\text{say.}$$

We may note that to estimate J_4 and J_5, it is sufficient to consider their expressions without the linear combination. By Lemma 2.4 we have

$$S_n^{(r)}(\phi(t)(f(t)-f_{2k+2,\delta}(t)),x)=\frac{(n+r-1)!}{n^{r-1}(n-1)!}\sum_{v=0}^{\infty}p_{n+r,v}(x)\times$$

$$\times\int_0^\infty q_{n,v+r}(t)\phi(t)(f^{(r)}(t)-f_{2k+2,\delta}^{(r)}(t))\,dt.$$

Hence

$$\left\|S_n^{(r)}(\phi(t)(f(t)-f_{2k+2,\delta}(t)),\cdot)\right\|_{C[a_1,b_1]}\le C_5\left\|f^{(r)}-f_{2k+2,\delta}^{(r)}\right\|_{C[a^*,b^*]}.$$

Now, for $x \in [a_1, b_1]$ and $t \in (0,\infty) \setminus [a^*, b^*]$, we can choose a $\delta_1 > 0$ satisfying $|t-x| \geq \delta_1$. Therefore by Lemma 2.3 and Schwarz inequality, we have

$$I = \left|S_n^{(r)}((1-\phi(t)(f(t)-f_{2k+2,\delta}(t)),x)\right| \leq$$

$$\leq n \sum_{\substack{2i+j\leq r \\ i,j>0}} n^i \frac{|\phi_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j \times$$

$$\times \int_0^{\infty} q_{n,v}(t)(1-\phi(t))|f(t)-f_{2k+2,\delta}(t)|\,dt \leq$$

$$\leq C_6 \|f\|_\gamma \, n \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j \int_{|t-x|\geq\delta} q_{n,v}(t)dt \leq$$

$$\leq C_6 \delta_1^{-2s} \|f\|_\gamma \, n \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \sum_{v=0}^{\infty} p_{n,v}(x)|v-nx|^j \times$$

$$\times \left(\int_0^{\infty} q_{n,v}(t)dt\right)^{1/2} \left(\int_0^{\infty} q_{n,v}(t)(t-x)^{4s}dt\right)^{1/2} \leq$$

$$\leq C_6 \delta_1^{-2s} \|f\|_\gamma \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i \left(\sum_{v=0}^{\infty} p_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \times$$

$$\times \left(n \sum_{v=0}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v}(t)(t-x)^{4s}dt\right)^{1/2}.$$

Hence, by Lemma 2.1. and Lemma 2.2, we have

$$I \geq C_7 \|f\|_\gamma \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^{(i+(j/2)-s)} \leq C_7 n^{-q} \|f\|_\gamma$$

where $q = (s-(r/2))$. Now choose $s > 0$ such that $q \geq k+1$. Then $I \leq C_7 n^{-(k+1)} \|f\|_\gamma$. Hence by property (iii) of Steklov mean, we get

$$J_1 \leq C_8 \left\|f^{(r)} - f_{2k+2,\delta}^{(r)}\right\|_{C[a^*,b^*]} + C_7 n^{-(k+1)} \|f\|_\gamma \leq$$

$$\leq C_9 \omega_{2k+2}(f^{(r)}, \delta, a, b) + C_7 n^{-(k+1)} \|f\|_\gamma .$$

Therefore, with $\delta = n^{-1/2}$, the theorem follows.

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