FASCICULI MATHEMATICI

Nr 31 2001

MAŁGORZATA MIGDA

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF NONLINEAR DELAY DIFFERENCE EQUATIONS

ABSTRACT: Asymptotic properties of the solutions of the difference equation of the form

$$\Delta(r_{n-1}\Delta x_{n-1}) + a_n f(x_{n-k}) = b_n$$

are studied.

KEY WORDS: difference equation, asymptotic behaviour.

1. INTRODUCTION

In this paper we are concerned with the asymptotic behaviour of solutions of the nonlinear difference equations with a forcing term

$$\text{(E)} \qquad \Delta(r_{n-1}\Delta x_{n-1}) + a_n f(x_{n-k}) = b_n, \qquad n = 1,2,\ldots$$

where k is a nonnegative integer, (r_n) is a sequence of positive real numbers, (a_n), (b_n) are real sequences ((a_n) cannot be eventually identically zero) and f is a real function. The forward difference operator Δ is defined as usual, i.e. $\Delta x_n = x_{n+1} - x_n$ for $n \in N$.

By a solution of (E) we mean a real sequence (x_n) which is defined for $n \geq -k$ and which satisfies equation (E) for all $n = 1,2,\ldots$, whereas by a generalized solution of (E) we mean such a sequence which satisfies (E) for all sufficiently large n.

In the last few years the study of the oscillatory and asymptotic behaviour of solutions of nonlinear difference equations has been subject of a great interest, see for example [2-3, 5-10], and the references contained therein. In most of the paper it is assumed that $uf(u) > 0$ for $u \neq 0$ and that f is superlinear or sublinear function. Here those assumptions are not needed.

2. MAIN RESULTS

Throughout this paper we will use the following notation:

$$R_n = \sum_{j=0}^{n-1} \frac{1}{r_j}.$$

THEOREM 1. *Suppose that*

(1) $$\sum_{n=1}^{\infty} R_n|a_n| < \infty, \qquad \sum_{n=1}^{\infty} R_n|b_n| < \infty,$$

holds and the function f is continuous. Then for every $c \in R$ *there exists a generalized solution* (x_n) *of the equation* (E) *such that*

(2) $$\lim_{n\to\infty} x_n = c.$$

PROOF. Let $c \in R$ and we choose a real number $a > 0$. Then there exists a constant $M > 1$ such that

$$|f(t)| < M \quad \text{for every} \quad t \in [c-a, c+a].$$

For $n \in N$ let

(3) $$\alpha_n = |a_n| + |b_n| \quad \text{and} \quad \beta_n = \frac{1}{r_n}\sum_{j=n+1}^{\infty} \alpha_j.$$

From (1), we have

$$\sum_{n=1}^{\infty} R_n\,|\,a_n\,| + \sum_{n=1}^{\infty} R_n\,|\,b_n\,| = \sum_{n=1}^{\infty} R_n\alpha_n = \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}\frac{1}{r_j}\right)\alpha_n =$$

$$= \alpha_1\frac{1}{r_0} + \alpha_2\left(\frac{1}{r_0}+\frac{1}{r_1}\right) + \ldots = \frac{1}{r_0}(\alpha_1+\alpha_2+\ldots) + \frac{1}{r_1}(\alpha_2+\alpha_3+\ldots) + \ldots =$$

$$= \sum_{n=1}^{\infty}\frac{1}{r_{n-1}}\sum_{j=n}^{\infty}\alpha_j = \sum_{n=1}^{\infty}\beta_{n-1}.$$

Therefore the series $\sum_{n=1}^{\infty}\beta_n$ is convergent, too. Let

(4) $$\rho_n = \sum_{j=n}^{\infty}\beta_j \quad \text{for} \quad n \in N.$$

Hence $\lim_{n\to\infty}\rho_n = 0$. There exists an index $n_0 \in N$ such that $M\rho_n < a$ for every $n \geq n_0 \geq k+1$.

Let l_∞ denote the Banach space of all real bounded sequences equipped with "sup" norm. Let

$$T = \{x \in l_\infty : x_1 = \ldots = x_{n_0-1} = c \quad \text{and} \quad |\,x_n - c\,| \leq M\rho_n \quad \text{for} \quad n \geq n_0\}.$$

Obviously, T is a convex and closed subset of the space l_∞. Let $\varepsilon > 0$. It is easy to construct a finite ε-net for the set T. Hence T is compact.

If $x \in T$ then $x_n \in [c-a, c+a]$ for each $n \in N$. Hence $|f(x_n)| < M$ for every $x \in T$, $n \in N$. Therefore, denoting

(5) $$u_n = \frac{1}{r_n}\sum_{j=n+1}^{\infty}[a_j f(x_{j-k}) - b_j], \qquad n = 1,2,\ldots$$

by (3) we have

(6) $$|u_n| \le \frac{M}{r_n}\sum_{j=n+1}^{\infty}|a_j| + \frac{1}{r_n}\sum_{j=n+1}^{\infty}|b_j| \le M\beta_n.$$

Since the series $\sum_{j=1}^{\infty}\beta_j$ is convergent, the series $\sum_{j=1}^{\infty}|u_j|$ is convergent, too. Now, we define the sequence $A(x)$ by

$$A(x)(n) = \begin{cases} c & \text{for} \quad n < n_0, \\ c - \sum_{j=n}^{\infty} u_j & \text{for} \quad n \ge n_0. \end{cases}$$

For $n \ge n_0$ by (4), (6) we have

$$|A(x)(n) - c| \le \sum_{j=n}^{\infty}|u_j| \le \sum_{j=n}^{\infty} M\beta_j = M\rho_n.$$

Hence $A(x) \in T$ for every $x \in T$, and we get a map $A: T \to T$. We will show that A is a continuous map. Let x, z be any two elements of the set T such that $\|x - z\| < \delta$. Then $|x_n - z_n| < \delta$ for every $n \in N$. Since the function f is uniformly continuous on the interval $[c-a, c+a]$ we get

(7) $$|f(x_n) - f(z_n)| < \varepsilon \ \text{ for each } \ n \in N.$$

Let us denote

(8) $$v_n = \frac{1}{r_n}\sum_{j=n+1}^{\infty}[a_j f(z_{j-k}) - b_j] \quad \text{for} \quad n \in N.$$

Then we have

$$\|A(x) - A(z)\| = \sup_{n \ge n_0}\left|\sum_{j=n}^{\infty} u_j - \sum_{j=n}^{\infty} v_j\right| \le \sum_{j=n_0}^{\infty}|u_j - v_j|.$$

By (5), (8) and (7) one yields

$$|u_j - v_j| \le \frac{1}{r_n}\sum_{i=j+1}^{\infty}|a_i||f(x_{i-k}) - f(z_{i-k})| \le \varepsilon\frac{1}{r_n}\sum_{i=j+1}^{\infty}|a_i| \le \varepsilon\beta_j.$$

Hence

$$\|A(x) - A(z)\| \le \sum_{j=n_0}^{\infty}\varepsilon\beta_j = \varepsilon\rho_{n_0}.$$

This shows A is a continuous map.

By Schauder fixed point theorem ([4]) there exists $z \in T$ such that $A(z) = z$. Then $z_n = c - \sum_{j=n}^{\infty} v_j$ for all $n \geq n_0$. Hence $\Delta z_{n-1} = v_{n-1}$ for all $n \geq n_0$ and

$$\Delta(r_{n-1}\Delta z_{n-1}) = r_n v_n - r_{n-1}v_{n-1} =$$

$$= \sum_{j=n+1}^{\infty} [a_j f(z_{j-k}) - b_j] - \sum_{j=n}^{\infty} [a_j f(z_{j-k}) - b_j] = -a_n f(z_{n-k}) + b_n$$

for every $n \geq n_0$. Therefore z is a generalized solution of (E). By convergence of the series $\sum_{j=1}^{\infty} v_j$ it follows that $\lim_{n\to\infty} z_n = c$. This completes the proof of the theorem.

Remark 1. If $k = 0$ then transforming (E) to the form

$$x_{n-1} = -\frac{r_n}{r_{n-1}} x_{n+1} + \left(1 + \frac{r_n}{r_{n-1}}\right) x_n - \frac{a_n}{r_{n-1}} f(x_n) + \frac{b_n}{r_{n-1}}$$

and putting $x_j = z_j$ for $j \geq n_0$ we can find $x_{n_0-1}, \dots, x_1$ succesively in a step by step fashion. The sequence thus obtained forms is an ordinary solution of (E) and posseses property (2).

Remark 2. Theorem 1 exstends Theorem 2 of [2].

Theorem 2. *If the function f is uniformly continuous and bounded, the series $\sum_{n=1}^{\infty} R_n|a_n|$, $\sum_{n=1}^{\infty} R_n|b_n|$ are convergent then for every $c, d \in R$ there exists a solution (x_n) of* (E) *which possesses the asymptotic behavior*

$$x_n = cR_n + d + \mathrm{o}(1).$$

Proof. Let $c, d \in R$. There exists a constant $M > 1$ such that $|f(t)| < M$ for each $t \in R$. Let the sequences (α_n), (β_n), (ρ_n), (u_n), (v_n) be the same as in the proof of Theorem 1. Let l be the space of all sequence $x : N \to R$ and let

$$T = \{x \in l : |x_n| \leq M\rho_n \quad \text{for all} \quad n \in N\},$$

$$S = \{x \in l_\infty : |x_n - (cR_n + d)| \leq M\rho_n \quad \text{for all} \quad n \in N\}.$$

Let us define a map $F : T \to S$ as follows

$$F(x)(n) = x_n + cR_n + d.$$

Obviously, the formula $d(x,z) = \sup\{|x_n - z_n| : n \in N\}$ defines a metric on the set S such that F is an isometry of the set T onto S. Clearly, T is a compact and convex subset of the space l_∞. The space S is homeomorphic to T. Hence by Schauder theorem every continuous map $A: S \to S$ has a fixed point.

Now, for $x \in S$ and $n \in N$ we define

$$A(x)(n) = cR_n + d - \sum_{j=n}^{\infty} u_j.$$

Then

$$|A(x)(n) - (cR_n + d)| = \left|\sum_{j=n}^{\infty} u_j\right| \le \sum_{j=n}^{\infty} |u_j| \le M\rho_n$$

for every $n \in N$. Hence $A(x) \in S$ and we get a map $A: S \to S$. Similar to the proof of Theorem 1 we can show that the map A is continuous. Hence there exists a sequence $x \in S$ such that $A(x) = x$. This is clealy a solution of equation (E). Furthermore, by convergence of the series $\sum_{j=1}^{\infty} u_j$ it follows that $x_n = cR_n + d + o(1)$.

REMARK 3. Theorem 2 extends Theorem 3 of [6].

THEOREM 3. *Suppose that the series* $\sum_{n=1}^{\infty} |a_n|$, $\sum_{n=1}^{\infty} |b_n|$ *are convergent and that f is a bounded function. Then every solution* (x_n) *of* (E) *satisfies the following:*

(i) *if* $\lim_{n\to\infty} R_n = \infty$ *then* $\lim_{n\to\infty} x_n / R_n = \mathrm{c} = \mathrm{constant}$,

(ii) *if* $\lim_{n\to\infty} r_n = g \ne 0$ *then* $\lim_{n\to\infty} \Delta x_n = \mathrm{d} = \mathrm{constant}$,

(iii) *if* $\lim_{n\to\infty} r_n = \infty$ *then* $\lim_{n\to\infty} \Delta x_n = 0$.

PROOF. Let (x_n) be a solution of (E). There exists a constant $M > 0$ such that $|f(t)| < M$ for every $t \in R$. Let $m > n$. Summing up both sides of (E) from $n+1$ to m we get

$$r_m \Delta x_m - r_n \Delta x_n = \sum_{j=n+1}^{m} b_j - \sum_{j=n+1}^{m} a_j f(x_{j-k}).$$

Therefore

$$|r_m \Delta x_m - r_n \Delta x_n| = \sum_{j=n+1}^{m} |b_j| + M \sum_{j=n+1}^{m} |a_j|.$$

Hence the sequence $(r_n \Delta x_n)$ is convergent. Let $\lim_{n\to\infty} r_n \Delta x_n = c$. Therefore, by Stolz's theorem (see for example [1] Theorem 1.7.9) we have

$$\lim_{n\to\infty}\frac{x_n}{R_n}=\lim_{n\to\infty}\frac{\Delta x_n}{R_{n+1}-R_n}=\lim_{n\to\infty}r_n\Delta x_n=c.$$

From convergence of the sequences $(r_n\Delta x_n)$ on one get (ii) and (iii).

REMARK 4. Theorem 3 extends Theorem 4 of [6].

REFERENCES

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York 1992.

[2] A. Drozdowicz, J. Popenda, Asymptotic behavior of the solutions of the second order difference equation, *Proc. Amer. Math. Soc.* 99(1)(1987), 135-140.

[3] A. Drozdowicz, J. Popenda, Asyptotic behavior of solutions of difference equations of second order, *Jurnal of Com. and Applied Math.* 47(1993), 141-149.

[4] R.E. Edwars, *Functional Analysis*, Holt, Rinehart and Winston Inc, New York 1965.

[5] J.W. Hooker, W.T. Patula, A second order nonlinear difference equation: oscillation and asymptotic behavior, *J. Math. Anal. Appl.* 91(1983), 9-29.

[6] J. Migda, M. Migda, Asymptotic properties of the solution of second order difference equation, *Archivum math.* 34(1998), 467-476.

[7] Z. Szafrański, B. Szmanda, Oscillatory properties of solutions of some difference systems, *Radovi Mat.* 6(1990), 205-214.

[8] Z. Szafrański, B. Szmanda, Oscillation Theorems for Some Nonlinear Difference Equations, *Appl. Math. Comput.* 83(1997), 43-52.

[9] B. Szmanda, Characterization of Oscillation of second Order Nonlinear Difference Equations, *Bull. Polish Acad. Sci. Math.* 34((1986), 133-141.

[10] E. Thandapani, L. Ramuppillai, Oscillatory and asymtotic behaviour of pertubed quasilinear second order difference equations, *Archivum Math.* 34(1998), 455-466.

(Institute of Mathematics, Poznań University of Technology, 60-965 Poznań, Piotrowo 3A, Poland (e-mail: mmigda@math.put.poznan.pl))

Received on 1999.06.02 and, in revised form, on 2000.04.27.