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OSCILLATORY PROPERTIES OF SOLUTIONS OF NEUTRAL DIFFERENTIAL SYSTEMS

ABSTRACT: The purpose of this paper is to obtain oscillation criterions for the neutral differential systems (S).

KEY WORDS: neutral differential system, oscillatory (nonoscillatory) solution.

1. INTRODUCTION

In this paper we consider the neutral differential system of the form

$$
\text{(S)}\qquad
\begin{aligned}
[y_1(t) - a(t)y_1(g(t))]' &= p(t)y_2(t),\\
y_i'(t) &= p(t)y_{i+1}(t), \quad i = 2,3,\ldots,n-2;\ n \geq 3\\
y_{n-1}'(t) &= p_{n-1}(t)y_n(t)\\
y_n'(t) &= p_n(t)y_1(h(t)), \quad i \in R_+ = [0,\infty).
\end{aligned}
$$

The following conditions are assumed to hold throughout this paper:

(a) $p: R_+ \to [K,\infty)$, $K > 0$ is a continuous function, $p_i : R_+ \to R_+$, $i = n-1, n$, are continuous functions and not identically zero in every neighbourhood of infinity,

$$\int^{\infty} p_{n-1}(t)dt = \infty;$$

(b) $a: R_+ \to R$ is a continuous function satisfying $|a(t)| \leq \lambda < 1$, where λ is a constant and $a(t)a(g(t)) \geq 0$ on R_+;

(c) $g: R_+ \to R$ is a continuous and increasing function, $g(t) < t$ on R_+ and $\lim_{t\to\infty} g(t) = \infty$;

(d) $h: R_+ \to R_+$ is a continuous function and $\lim_{t\to\infty} h(t) = \infty$.

The asymptotic properties of solutions of systems with deviating arguments are studied for example in the papers [1, 5-8, 10-12, 14, 15]. As far as is know to the author, the oscillatory theory of systems of neutral differential equations is studied only in the papers [2-4, 9, 13, 16]. The purpose of this paper is to obtain oscillation criterions for the system (S). This paper is generalization of the results obtained in the paper [15].

Let $t_0 \geq 0$. Denote

$$\widetilde{t_0} = \min\left\{g(t_0), \inf_{t\geq t_0} h(t)\right\}.$$

A function $y=(y_1,\dots,y_n)$ is solution of the system (S) if there exists a $t_0 \geq 0$ such that y is continuous on $[\widetilde{t_0},\infty)$, $y_1(t)-a(t)y_1(g(t)), y_i(t)$, $i=2,\dots,n$ are continuously differentiable on $[t_0,\infty)$ and y satisfies (S) on $[t_0,\infty)$.

Denote by W the set of all solutions $y=(y_1,\dots y_n)$ of the system (S) which exist on some ray $[T_y,\infty)\subset R_+$ and satisfy

$$\sup\left\{\sum_{i=1}^{n} |y_i(t)| : t\geq T\right\}>0 \quad \text{for any} \quad T\geq T_y.$$

A solution $y\in W$ is nonoscillatory if there exists a $T^*\geq T_y$ such that its every component is different from zero for all $t\geq T^*$. Otherwise a solution $y\in W$ is said to be oscillatory.

The following notation will be used throughout this paper: Let $r(t): R_+\to R_+$ be a continuous and nondecreasing function such that $r(t)\leq \min\{t,h(t)\}$ for $t\in R_+$ and $\lim_{t\to\infty} r(t)=\infty$.

$$\gamma(t)=\sup\{s\geq 0, r(s)\leq t\}, \qquad t>0.$$

Let $t,s\in R_+$, $k\in\{0,1,\dots,n-2\}$. We define: $I_0\equiv 1$,

$$I_k(t,s;p) = \int_s^t p(x)\, I_{k-1}(x,s;p)\,dx;$$

$$P_j(t)=I_j(t,0;p), \quad j\in\{0,1,\dots,n-2\}.$$

It is not difficult to verify that the following inequalities hold:

(1_m^i) $$P_m(t)\leq (P_1(t))^i P_{m-i}(t), \quad \text{for} \quad t\in R_+,$$
$$m\in\{1,2,\dots,n-2\}, \quad i\in\{0,1,\dots,m\}.$$

For any $y_1(t)$ we define $z_1(t)$ by

(2) $$z_1(t)=y_1(t)-a(t)y_1(g(t)).$$

2. Some basic lemmas

The following lemmas will be useful in the proofs of the main results.

Lemma 1. ([9, Lemma 1]). *Let $y \in W$ be a solution of the system* (S) *with $y_1(t) \neq 0$ on $[t_0, \infty)$, $t_0 \geq 0$. Then y is nonoscillatory and $z_1(t)$, $y_2(t)$, ..., $y_n(t)$* are monotone on some ray $[T, \infty)$, $T \geq t_0$.

Lemma 2. ([9, Lemma 2]). *Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of the system* (S) *and let $\lim_{t \to \infty} |z_1(t)| = L_1$, $\lim_{t \to \infty} |y_i(t)| = L_i$, $i = 2, \ldots, n$. Then*

(3) *if* $k \geq 2, L_k > 0$ *implies* $\lim_{t \to \infty} |y_1(t)| = L_1 = L_2 = \ldots = L_{k-1} = \infty$;

(4) *if* $1 \leq k < n$, $L_k < \infty$ *implies* $L_{k+1} = L_{k+2} = \ldots = L_n = 0$.

Lemma 3. ([9, Lemma 4]). *Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of the system* (S) *on $[t_0, \infty)$, $t_0 \geq 0$. Then there exist an integer $l \in \{1, 2, \ldots, n\}$ with $n + l$ odd or $n = l$, and a $t_1 \geq t_0$ such that for $t \geq t_1$ either*

$$(5_l) \qquad \begin{aligned} &z_1(t) y_1(t) > 0 \\ &y_i(t) y_1(t) > 0, \quad i = 1, 2, \ldots, l \\ &(-1)^{l+i} y_i(t) y_1(t) > 0, \quad i = l, l+1, \ldots, n \end{aligned}$$

or

$$(6) \qquad \begin{aligned} &z_1(t) y_1(t) < 0 \\ &(-1)^i y_i(t) y_1(t) > 0, \quad i = 2, \ldots, n, \quad \textit{where } n \textit{ is odd.} \end{aligned}$$

Remark. The case $z_1(t) y_1(t) < 0$ on $[t_1, \infty)$ can occur only if $a(t) > 0$ on $[t_1, \infty)$ and n is odd.

We denote by N_l^+ or N_2^- the set of all nonoscillatory solutions of (S) which satisfy (5_l) or (6) respectively. Denote by N the set of all nonoscillatory solutions of (S). Then Lemma 3 the following classification holds for $t \in [t_1, \infty)$:

$$(7) \qquad \begin{aligned} &N = N_1^+ \cup N_3^+ \cup \ldots \cup N_{n-1}^+ \cup N_n^+ \quad \text{for } n \text{ even}, \\ &N = N_2^+ \cup N_4^+ \cup \ldots \cup N_{n-1}^+ \cup N_n^+ \cup N_2^- \quad \text{for } n \text{ odd}. \end{aligned}$$

Lemma 4. ([9, Lemma 5]). I). *Let* $y \in N_l^+$ *on* $[t_1,\infty)$, $l \geq 2$. *Then there exists a* $t_2 \geq t_1$ *such that*

$$|y_1(t)| \geq (1-\lambda)|z_1(t)| \quad for \quad t \geq t_2. \tag{8}$$

II) *Let* $y \in N_1^+$ *on* $[t_1,\infty)$.

(i) *If* $\lim_{t\to\infty} |z_1(t)| = L_1 > 0$, *then there exists an* $\alpha : 0 < \alpha < 1$ *such that*

$$|y_1(t)| \geq \alpha |z_1(t)| \quad for \quad t \geq t_2. \tag{9}$$

(ii) *If* $\lim_{t\to\infty} z_1(t) = 0$ then $\liminf_{t\to\infty} |y_1(t)| = 0$, $\lim_{t\to\infty} y_i(t) = 0$, $i = 2,\ldots,n$.

Lemma 5. ([9, Lemma 6]). *Let* $y \in N_2^-$ *on* $[t_1,\infty)$. *Then*

$$\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} y_i(t) = 0, \quad i = 1,\ldots,n.$$

Lemma 6. *Let* $y \in W$ *be a solution of* (S) *on* $[t_0,\infty)$, $t_0 \geq 0$. *Then the following relations hold:*

$$z_1(t) = z_1(s) + \int_s^t p(x) y_2(x)\,dx; \tag{$10_{1,1}$}$$

$$z_1(t) + \sum_{j=2}^{k} (-1)^{1+j} y_j(t) P_{j-1}(t) = z_1(s) + \sum_{j=2}^{k} (-1)^{1+j} y_j(s) P_{j-1}(s) + \\ + (-1)^{1+k} \int_s^t P_{k-1}(x) p_k(x) y_{k+1}(x)\,dx, \tag{$10_{1,k}$}$$

for $k = 2,3,\ldots,n-1$, $t,s \in [t_0,\infty)$, where $p_k(x) \equiv p(x)$ for $k = 2,3,\ldots,n-2$;

$$\sum_{j=i}^{k} (-1)^{i+j} y_j(t) P_{j-i}(t) = \sum_{j=i}^{k} (-1)^{i+j} y_j(s) P_{j-i}(s) + \\ + (-1)^{i+k} \int_s^t P_{k-i}(x) p_k(x) y_{k+1}(x)\,dx \tag{$10_{i,k}$}$$

for $i = 2,3,\ldots,n-1$, $k = i, i+1,\ldots,n-1$, $t,s \in [t_0,\infty)$ *where* $p_k(x) \equiv p(x)$ *for* $k = 2,3,\ldots,n-2$.

PROOF. Integrating i-th equation of (S) from s to t we get $(10_{i,i})$, $i = 1,2,\dots,n-1$. Suppose that $(10_{i,k-1})$ is true for $i = 1,2,\dots,n-1$, $k \in \{i+1,\dots,n-1\}$. Integrating

$$(-1)^{i+k}\int_s^t P_{k-i}(x)p_k(x)y_{k+1}(x)dx$$

by parts and then using $(10_{i,k-1})$ we get $(10_{i,k})$.

LEMMA 7. *Let* $y \in N_l^+$ *on* $[t_1,\infty)$, $l \in \{2,3,\dots,n-1\}$. *In addition let*

$$\int^{\infty} P_{n-l}(x)p_{n-1}(x)\,|y_n(x)|\,dx = \infty. \tag{11}$$

Then there exists a $t_3 \geq t_2$ *such that*

(12) $\dfrac{|z_1(t)|}{(P_1(t))^{l-1}}$ *is nonincreasing function on* $[t_3,\infty)$;

$$|z_1(t)| \geq \frac{P_{n-l}(t)}{(l-1)!(P_1(T))^{n-2l+1}}\int_t^{\infty} P_{n-l-1}(x)p_{n-1}(x)\,|y_n(x)|\,dx, \quad t \geq t_3. \tag{13}$$

PROOF. I) Let $l = 2$. From equation $(10_{1,n-1})$ with regard to (11) and (5_2) we obtain

$$\lim_{t\to\infty} |z_1(t) - y_2(t)P_1(t)| = \infty,$$

$$|z_1(t)| \geq |y_2(t)|\,P_1(t), \quad t \geq t_3 \tag{14}$$

and

$$|z_1(t)| \geq \left|\sum_{j=2}^{n-1}(-1)^{j+2}y_j(t)P_{j-1}(t)\right|, \quad t \geq t_3, \tag{15}$$

for sufficiently large $t_3 \geq t_2$.

Form (14) we obtain (12) for $l = 2$. Using (5_2) and (1_{j-2}^{j-2}) from $(10_{2,n-1})$ we get

$$\left|\sum_{j=2}^{n-1}(-1)^{j+2}y_j(t)P_{j-2}(t)\right| \geq \int_t^{\infty} P_{n-3}(x)p_{n-1}(x)\,|y_n(x)|dx, \quad t \geq t_3$$

and

$$(16)\qquad \left|\sum_{j=2}^{n-1}(-1)^{j+2}y_j(t)(P_1(t))^{j-2}\right| \ge \int_t^\infty P_{n-3}(x)p_{n-1}(x)|y_n(x)|dx, \quad t\ge t_3.$$

From (15) with regard to (1_{n-2}^{n-j-1}) we have

$$(17)\qquad |z_1(t)|\ge\left|\sum_{j=2}^{n-1}(-1)^{j+2}y_j(t)\frac{P_{n-2}(t)}{(P_1(t))^{n-j-1}}\right|, \quad t\ge t_3.$$

Multiplying (16) by $P_{n-2}(t)/(P_1(t))^{n-3}$ and then using (17) we get (13) for $l=2$.

II) Let $l\in\{3,4,\dots,n-1\}$. From equation $(10_{l-1,n-1})$ with regard to (11) and (5_l) we obtain

$$(18)\qquad \lim_{t\to\infty}|y_{l-1}(t)-y_l(t)P_1(t)|=\infty$$

and

$$(19)\qquad |y_{l-1}(t)|\ge\left|\sum_{j=l}^{n-1}(-1)^{j+l}y_j(t)P_{j-l+1}(t)\right|, \quad t\ge\tilde{t}_2,$$

for sufficiently large $\tilde{t}_2\ge t_2$.

Using (5_l) and (1_{j-l}^{j-l}) form $(10_{l,n-1})$ we get

$$\left|\sum_{j=1}^{n-1}(-1)^{j+l}y_j(t)P_{j-l}(t)\right| \ge \int_t^\infty P_{n-l-1}(x)p_{n-1}(x)|y_n(x)|dx, \quad t\ge\tilde{t}_2$$

and

$$(20)\qquad \left|\sum_{j=l}^{n-1}(-1)^{j+l}y_j(t)(P_1(t))^{j-l}\right| \ge \int_t^\infty P_{n-l-1}(x)p_{n-1}(x)|y_n(x)|dx, \quad t\ge\tilde{t}_2.$$

From (19) with regard to (1_{n-l}^{n-j-1}) we have

$$(21)\qquad |y_{l-1}(t)|\ge\left|\sum_{j=l}^{n-1}(-1)^{j+l}y_j(t)\frac{P_{n-l}(t)}{(P_1(t))^{n-j-1}}\right|, \quad t\ge\tilde{t}_2.$$

Multiplying (20) by $P_{n-l}(t)/(P_1(t))^{n-l-1}$ and then using (21) we get

$$(22) \qquad |y_{l-1}(t)| \geq \frac{P_{n-l}(t)}{(P_1(t))^{n-l-1}} \int_t^\infty P_{n-l-1}(x) p_{n-1}(x) |y_n(x)| dx, \qquad t \geq \tilde{t}_2.$$

Denote

$$\rho_k = |k y_{l-k}(t) - y_{l-k+1}(t) P_1(t)|, \quad k = 1,2,\ldots,l-2,$$
$$\rho_{l-1}(t) = |(l-1) z_1(t) - y_2(t) P_1(t)|.$$

It is easy to prove that $(\rho_{k+1}(t))' = p(t)\rho_k(t)$, $t \geq \tilde{t}_2$, $k = 1,2,\ldots,l-2$. Using (18) and (*a*) we have $\lim\limits_{t\to\infty} \rho_i(t) = \infty$, $i = 1,2,\ldots,l-1$ and

$$(23_k) \qquad k|y_{l-k}(t)| \geq |y_{l-k+1}(t)| P_1(t), \quad t \geq t_k^*, \quad k = 1,2,\ldots,l-2,$$

$$(23_{l-1}) \qquad (l-1)|z_1(t)| \geq |y_2(t)| P_1(t), \quad t \geq t_{l-1}^*,$$

where the points t_i^*, $i = 1,2,\ldots,l-1$ we can choose such that $t_{l-1}^* \geq t_{l-2}^* \geq \geq \ldots \geq t_2^* \geq t_1^* \geq \tilde{t}$. From (23_{l-1}) we obtain (12) and combining (22) with (23_i), $i = 2,3,\ldots,l-1$ we get (13) where $t_3 = t_{l-1}^*$.

Lemma 8. *Let* $y \in W$ *be a nonoscillatory solution of* (S) *on* $[t_0,\infty)$, $t_0 \geq 0$. *Then there exists a* $t_4 \geq t_3$ *such that*

i) *If* $y \in N_1^+$ *on* $[t_1,\infty)$ *then*

$$(24) \qquad |z_1(t)| \geq \int_t^\infty p(x_1) \int_{x_1}^\infty p(x_2) \ldots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \times$$
$$\times \int_{x_{n-1}}^\infty p_n(x_n) |y_1(h(x_n))| \, dx_n dx_{n-1} \ldots dx_1, \qquad t \geq t_4.$$

ii) *If* $y \in N_l^+$ *on* $[t_1,\infty)$, $l \in \{2,3,\ldots,n-1\}$ *then*

$$(25) \qquad |y_n(t)| \geq (1-\lambda) \int_t^\infty p_n(x) |z_1(r(x))| dx, \qquad t \geq t_4.$$

iii) *If* $y \in N_n^+$ *on* $[t_1,\infty)$ *then*

$$(26) \qquad |y_n(t)| \geq (1-\lambda) \int_s^t p_n(x_n) \int_{r(s)}^{r(x_n)} p(x_1) \int_{r(s)}^{x_1} p(x_2) \times$$

$$\times \ldots \int_{r(s)}^{x_{n-2}} p_{n-1}(x_{n-1})\,|y_n(x_{n-1})|\,dx_{n-1}dx_{n-2}\ldots dx_1 dx_n, \qquad t \ge s \ge t_4.$$

PROOF. i) In this case the functions $|z_1(t)|$, $|y_i(t)|$, $i = 2,3,\ldots,n$ are nonincreasing on $[\gamma(t_1),\infty)$. Integrating all equations of (S) using (5_1) and then by their combination we have (24).

ii) In this case the function $|y_n(t)|$ is nonincreasing on $[\gamma(t_1),\infty)$. Integrating the last equation of (S) and then using (5_l), (8) and the monotonicity of $|z_1(t)|$ we have (25).

iii) In this case the functions $|z_1(t)|$, $|y_i(t)|$, $i = 2,3,\ldots,n$ are nondecreasing on $[\gamma(t_1),\infty)$. Integrating all equations of (S) and then by their combination with help (5_n) and (8) we have (26).

3. OSCILLATION THEOREMS

THEOREM 1. *Let n be odd and the assumptions*

$$\limsup_{t\to\infty} \left\{ \frac{P_{n-2}(r(t))}{(P_1(r(t)))^{n-2}} \left(\int_{r(t)}^{t} P_{n-2}(r(x)) p_{n-1}(x) \int_x^t p_n(s)\,ds\,dx + \right.\right.$$
$$\left.\left. + P_{n-2}(r(t)) \int_t^\infty p_{n-1}(x) \int_x^\infty p_n(s)\,ds\,dx \right)\right\} > \frac{(n-2)!}{1-\lambda} \tag{27}$$

and

$$\int_s^\infty p_n(x_n) \int_{r(s)}^{r(x_n)} p(x_1) \int_{r(s)}^{x_1} p(x_2) \times \ldots \times \tag{28}$$
$$\times \int_{r(s)}^{x_{n-2}} p_{n-1}(x_{n-1})\,dx_{n-1}\ldots dx_2\,dx_1\,dx_n = \infty$$

hold. Then every solution $y \in W$ *of* (S) *is either oscillatory or* $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1,2,\ldots,n$ *or* $\lim_{t\to\infty} y_i(t) = 0$, $i = 1,2,\ldots,n$.

PROOF. Let $y \in W$ be a nonoscillatory solution of (S). Using (7) we get

$$y \in N_2^+ \cup N_4^+ \cup \ldots \cup N_{n-1}^+ \cup N_n^+ \cup N_2^- \quad \text{on } [t_1,\infty).$$

A) Let $y \in N_l^+$ on $[t_1,\infty)$, $l \in \{2,4,\dots,n-1\}$. We shall prove that (11) holds. Denote

$$(29) \qquad \int_{t_4}^{\infty} P_{n-l}(x) p_{n-1}(x) \int_{x}^{\infty} p_n(s) ds dx = \infty.$$

From (11) with regard to (25) and the monotonicity of $|z_1(t)|$ we get

$$\int_{t_4}^{\infty} P_{n-l}(x) p_{n-1}(x) |y_n(x)| dx \geq K_1 \int_{t_4}^{\infty} P_{n-l}(x) p_{n-1}(x) \int_{x}^{\infty} p_n(s) ds\, dx,$$

where $K_1 = (1-\lambda)\,|z_1(r(t_4))| > 0$. The last inequality implies that (29) yields (11). We shall prove that (27) implies (29). Suppose the contrary:

$$\int_{t_4}^{\infty} P_{n-2}(x) p_{n-1}(x) \int_{x}^{\infty} p_n(s) ds\, dx < \infty.$$

Then there exists a $t_5 \geq t_4$ such that

$$(30) \qquad \int_{r(t)}^{t} P_{n-2}(x) p_{n-1}(x) \int_{x}^{t} p_n(s) ds\, dx + \int_{t}^{\infty} P_{n-2}(x) p_{n-1}(x) \int_{x}^{\infty} p_n(s) ds\, dx \leq \frac{(n-2)!}{1-\lambda}, \qquad t \geq t_5.$$

From (30) with regard to (1_{n-2}^{n-2}) and the monotonicity of $P_{n-2}(x)$ we have

$$\frac{P_{n-2}(r(t))}{(P_1(r(t)))^{n-2}} \left(\int_{r(t)}^{t} P_{n-2}(r(x)) p_{n-1}(x) \int_{x}^{t} p_n(s) ds\, dx + P_{n-2}(r(t)) \int_{t}^{\infty} p_{n-1}(x) \int_{x}^{\infty} p_n(s) ds dx \right) \leq \frac{(n-2)!}{1-\lambda}, \qquad t \geq t_5,$$

which contradicts (27). The assumption (11) of Lemma 7 is satisfied. From (13) and (25) we get

$$(31) \quad |z_1(r(t))| \geq \frac{(1-\lambda) P_{n-1}(r(t))}{(l-1)!\,(P_1(r(t)))^{n-2l+1}} \left\{ \int_{r(t)}^{t} p_n(s) |z_1(r(s))| \times \right.$$

$$\times \int_{r(t)}^{s} P_{n-l-1}(x)p_{n-1}(x)\,dx\,ds + \int_{t}^{\infty} p_n(s)|z_1(r(s))| \int_{r(t)}^{s} P_{n-l-1}(x)p_{n-1}(x)\,dx\,ds \Bigg\},$$

$t \ge t_6$, $t_6 = \max\{\gamma(\gamma(t_3)), t_4\}$. From (12) for $s \in [r(t), t]$ we obtain

$$(32) \qquad |z_1(r(s))| \ge \frac{|z_1(r(t))|\,(P_1(r(s)))^{l-1}}{(P_1(r(t)))^{l-1}}, \quad \text{for} \quad t \ge \gamma(\gamma(t_3)).$$

From (31) in view of (32) and the monotonicity of $|z_1(t)|$ we derive

$$\frac{(l-1)!\,|z_1(r(t))|\,(P_1(r(t)))^{n-2l+1}}{P_{n-l}(r(t))} \ge \frac{(1-\lambda)\,|z_1(r(t))|}{(P_1(r(t)))^{l-1}} \times$$

$$\times \Bigg\{ \int_{r(t)}^{t} p_n(t)(P_1(r(s)))^{l-1} \int_{r(t)}^{s} P_{n-l-1}(x)p_{n-1}(x)\,dx\,ds +$$

$$+ (P_1(r(t)))^{l-1} \int_{t}^{\infty} p_n(s) \int_{r(t)}^{s} P_{n-l-1}(x)p_{n-1}(x)\,dx\,ds \Bigg\}, \qquad t \ge t_6.$$

Using (1_{n-1}^{l-1}), (1_{n-2}^{l-2}) and the monotonicity of $P_{n-l-1}(x)$ and $P_1(r(s))$ from the last inequality we obtain

$$(n-2)! \ge (1-\lambda)\frac{P_{n-2}(r(t))}{(P_1(r(t)))^{n-2}} \Bigg\{ \int_{r(t)}^{t} P_{n-2}(r(x))p_{n-1}(x) \int_{x}^{t} p_n(s)\,ds\,dx +$$

$$+ P_{n-2}(r(t)) \int_{t}^{\infty} p_{n-1}(x) \int_{x}^{\infty} p_n(s)\,ds\,dx \Bigg\}, \qquad t \ge t_6,$$

which contradicts (27) and $N_2^+ \cup N_4^+ \cup \ldots \cup N_{n-1}^+ = 0$.

B) Let $y \in N_n^+$ on $[t_1, \infty)$. In this case the functions $|z_1(t)|$, $|y_i(t)|$, $i = 2,3,\ldots,n$ are nondecreasing on $[\gamma(t_1), \infty)$ and $\lim_{t\to\infty} |y_n(t)| = L_n > 0$. From (3) we have $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1,2,\ldots,n-1$. We shall prove that $L_n = \infty$. In view of (26) we get

$$|y_n(t)| \ge (1-\lambda)|y_n(r(s))| \int_{s}^{t} p_n(x_n) \int_{r(s)}^{x_1} p(x_2) \times \ldots$$

$$\ldots \times \int_{r(s)}^{x_{n-2}} p_{n-1}(x_{n-1})\, dx_{n-1} \ldots dx_1\, dx_n, \qquad t \geq s \geq t_4 .$$

The last inequality and (28) yield $L_n = \infty$.

C) Let $y \in N_2^-$ on $[t_1, \infty)$. Then using Lemma 5 we have $\lim_{t\to\infty} y_i(t) = 0$, $i = 1,2,\ldots,n$. The proof of Theorem 1 is complete.

THEOREM 2. *Let n be even and the condition* (28) *holds. In addition*

$$(33)\quad \limsup_{t\to\infty} \left\{ \frac{P_{n-3}(r(t))}{(P_1(r(t)))^{n-3}} \left(\int_{r(t)}^{t} P_{n-3}(r(x) p_{n-1}(x) \int_{x}^{t} p_n(s)\, ds\, dx + \right.\right.$$
$$\left.\left. + P_{n-3}(r(t)) \int_{t}^{\infty} p_{n-1}(x) \int_{x}^{\infty} p_n(s)\, ds\, dx \right) \right\} > \frac{(n-2)!}{1-\lambda}$$

and

$$(34)\qquad \int_{t}^{\infty} p(x_1) \int_{x_1}^{\infty} p(x_2) \ldots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n)\, dx_n\, dx_{n-1} \ldots dx_1 = \infty$$

hold. Then every solution $y \in W$ *of* (S) *is either oscillatory or* $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 2,3,\ldots,n$ *or* $\liminf_{t\to\infty} |y_1(t)| = 0$, $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} y_i(t) = 0$, $i = 2,3,\ldots,n$.

PROOF. Let $y \in W$ be a nonoscillatory solution of (S). Using (7) we get

$$y \in N_1^+ \cup N_3^+ \cup \ldots \cup N_{n-1}^+ \cup N_n^+ \quad \text{on} \quad [t_1, \infty).$$

A) Let $y \in N_l^+$ on $[t_1, \infty)$, $l \in \{3,5,\ldots,n-1\}$. By arguments similar to those as in A) of the proof of Theorem 1, we prove that this case is impossible.

B) Let $y \in N_n^+$ on $[t_1, \infty)$. Analogously as in the case B) of the proof of Theorem 1 we can show that $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1,2,\ldots,n$.

C) Let $y \in N_1^+$ on $[t_1, \infty)$. Then the functions $|z_1(t)|$, $|y_i(t)|$, $i = 2,3,\ldots,n$ are nonincreasing on $[\gamma(t_1), \infty)$ and $\lim_{t\to\infty} |z_1(t)| = L_1 < \infty$. From (4) we obtain $\lim_{t\to\infty} y_i(t) = 0$, $i = 2,3,\ldots,n$. We shall prove $L_1 = 0$.

Let $L_1 > 0$. From (24) and (9) we have

$$|z_1(t)| \geq \alpha L_1 \int_t^\infty p(x_1) \int_{x_1}^\infty p(x_2) \dots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \int_{x_{n-1}}^\infty p_n(x_n) dx_n \, dx_{n-1} \dots dx_1,$$

for sufficiently large t, which contradicts (34) and $L_1 = 0$. From Lemma 4 we have $\liminf_{t\to\infty} y_1(t) = 0$. The proof of Theorem 2 is complete.

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