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APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY MODIFIED SZASZ-MIRAKYAN OPERATORS

ABSTRACT: We consider certain modified Szasz-Mirakyan operators in the exponential weighted spaces of continuous functions of two variables. We prove theorems on the degree of approximation, the Voronovskaya type theorem and we study some differential properties of these operators.

The similar results for functions of one variable were given in [1] and [2].

KEY WORDS: Szasz-Mirakyan operator, approximation theorem, function of two variables.

1. PRELIMINARIES

1.1. Let as in [1] for a fixed $p > 0$

(1) $$v_p(x) := \exp(-px), \qquad x \in R_0 := [0,+\infty),$$

and let for fixed $p,q > 0$

(2) $$v_{p,q}(x,y) := v_p(x)v_q(y), \quad (x,y) \in R_0^2 := R_0 \times R_0.$$

Denote by $C_{p,q}$ the set of all real – valued functions f continuous on R_0^2 for which $v_{p,q}f$ is uniformly continuous and bounded on R_0^2 and the norm is defined by

(3) $$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in R_0^2} v_{p,q}(x,y)|f(x,y)|.$$

The modulus of continuity of $f \in C_{p,q}$ we define by

(4) $$\omega(f, C_{p,q}; s,t) := \sup_{\substack{0\le u\le s \\ 0\le v\le t}} \|\Delta_{u,v} f(\cdot,\cdot)\|_{p,q}, \qquad t,s \ge 0,$$

where $\Delta_{u,v} f(x,y) := f(x+u, y+v) - f(x,y)$. Let $C_{p,q}^m$, $m \in N := \{1,2,...\}$ and $p,q > 0$, be the class of all functions $f \in C_{p,q}$ which the partial derivatives of the order $k \le m$ belong also to $C_{p,q}$.

1.2. In given space $C_{p,q}$ we define the following modified Szasz-Mirakyan operators

(5) $$S_{m,n}(f;x,y) \equiv S_{m,n}(f;a_m,b_m,c_n,d_n;x,y) :=$$

$$:= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varphi_j(a_m x)\varphi_k(c_n y) f\left(\frac{j}{b_m}, \frac{k}{d_n}\right),$$

for $(x,y) \in R_0^2$ and $m,n \in N$, where

(6) $$\varphi_i(t) := e^{-t}\frac{t^i}{i!} \qquad \text{for} \quad t \in R_0, \quad i \in N_0 := N \cup \{0\},$$

and $(a_n)_1^\infty, (b_n)_1^\infty, (c_n)_1^\infty, (d_n)_1^\infty$ are given increasing and unbounded sequences of positive numbers and such that

(i) $$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right), \qquad \frac{c_n}{d_n} = 1 + o\left(\frac{1}{d_n}\right) \qquad \text{as} \quad n \to \infty,$$

(ii) $$\frac{a_{n+1}}{b_{n+1}} \le \frac{a_n}{b_n}, \qquad \frac{c_{n+1}}{d_{n+1}} \le \frac{c_n}{d_n} \qquad \text{for all} \quad n \in N.$$

If $a_n = b_n = c_n = d_n = n$ for all $n \in N$, then $S_{m,n}(f)$ defined by (5) is classical Szasz-Mirakyan operator, considered in [3] for continuous and bounded functions.

In [2] were examined the operators

(7) $$S_n(f;x) \equiv S_n(f;a_n,b_n;x) := \sum_{k=0}^{\infty} \varphi_k(a_n x) f\left(\frac{k}{b_n}\right), \quad x \in R_0, \quad n \in N,$$

for functions f of one variable, belonging to exponential weighted space.

In this paper we shall prove similarly results for operators $S_{m,n}(f)$. The basic properties of $S_{m,n}$ we shall give in Section 2 and main theorems we shall prove in Section 3.

In our paper we shall denote by $M_k(\alpha,\beta)$, $k = 1,2,...$, the suitable positive constants depending only on indicated parameters α, β.

2. AUXILIARY RESULTS

2.1. From (5) and (6) we deduce that $S_{m,n}(f)$ is well-defined in every space $C_{p,q}$, $p,q>0$. In particular

(8) $$S_{m,n}(1;x,y)=1, \qquad (x,y)\in R_0^2, \quad m,n\in N,$$

and if $f\in C_{p,q}$ and $f(x,y)=f_1(x)f_2(y)$ for all $(x,y)\in R_0^2$, then

(9) $$S_{m,n}(f;a_m,b_m,c_n,d_n;x,y)=S_m(f_1;a_m,b_m;x)S_n(f_2;c_n,d_n;y)$$

for all $x,y\in R_0$ and $m,n\in N$. Moreover from (7) we get

(10) $$S_n(1;a_n,b_n;x)=1, \qquad x\in R_0, \ n\in N.$$

2.2. In [2] were proved the following properties of $S_n(f;a_n,b_n;\cdot)$ definied by (7):

LEMMA 1. *Let $p>0$ be given number. Then for all $x\in R_0$ and $n\in N$ we have*

(11) $$S_n(t-x;x)=\left(\frac{a_n}{b_n}-1\right)x, \quad S_n((t-x)^2;x)=\left(\frac{a_n}{b_n}-1\right)^2x^2+\frac{a_nx}{b_n^2},$$

(12) $$S_n(e^{pt};x)=e^{p_nx}, \qquad S_n(te^{pt};x)=\frac{a_nx}{b_n}e^{p_nx+p/b_n},$$

(12) $$S_n((t-x)^2e^{pt};x)=$$
$$=e^{p_nx}\left\{\frac{x^2}{b_n^2}\left[e^{p/b_n}(a_n-b_n)+b_n(e^{p/b_n}-1)\right]^2+\frac{a_nx}{b_n^2}e^{p/b_n}\right\},$$

where

(14) $$p_n:=a_n(e^{p/b_n}-1) \qquad \textit{for } n\in N.$$

REMARK. From (14) and the properties of sequences $(a_n)_1^\infty$ and $(b_n)_1^\infty$ given in Section 1.2 we deduce that $(p_n)_1^\infty$ is decreasing sequence and

(15) $$p_n > \frac{a_n}{b_n} p \geq p \quad \text{and} \quad p_n < \frac{a_n}{b_n} p e^{p/b_n} \leq \frac{a_1}{b_1} p e^{p/b_n},$$

for all $n \in N$.

LEMMA 2. *Suppose that two given sequences* $(a_n)_1^\infty$ and $(b_n)_1^\infty$ *have the properties as in definition* (5) *and three given numbers satisfy the conditions:* $p > 0$, $r > (a_1/b_1)p$ *and* $n_0 \equiv n_0(p, r, a_1, b_1)$ *is a natural number such that*

(16) $$b_{n_0} > p\left(\ln \frac{rb_1}{pa_1}\right)^{-1}.$$

Then there exists a positive constant $M_1 \equiv M_1(p, r, a_1, b_1, n_0)$ *such that for* S_n *definied by* (7) *we have*

(17) $$v_r(x) S_n\left((t-x)^2 / v_p(t); x\right) \leq M_1 \frac{x}{b_n} \qquad \textit{for all} \quad x \in R_0,\ n > n_0.$$

Moreover,

(18) $$\sup_{x \in R_0} v_r(x) S_n\left(1/v_p(t); x\right) \leq 1 \qquad \textit{for all} \quad n > n_0.$$

LEMMA 3. *For every* $x \in R_0$ *we have*

$$\lim_{n \to \infty} b_n S_n(t - x; x) = 0, \qquad \lim_{n \to \infty} b_n S_n((t-x)^2; x) = x,$$

$$\lim_{n \to \infty} b_n^2 S_n((t-x)^4; x) = 3x^2.$$

2.3. Applying Lemma 2, we shall prove the basic property of $S_{m,n}(f)$.

LEMMA 4. *Assume that given sequences* $(a_n)_1^\infty$, $(b_n)_1^\infty$, $(c_n)_1^\infty$, $(d_n)_1^\infty$ *have the properties as in definition* (5) *and given numbers satisfy the conditions:* $p > 0$, $q > 0$, $r > (a_1/b_1)p$, $s > (c_1/d_1)q$ *and* $m_0 \equiv m_0(p, r, a_1, b_1)$, $n_0 \equiv n_0(q, s, c_1, d_1)$ *are natural numbers such that*

(19) $$b_{m_0} > p\left(\ln \frac{rb_1}{pa_1}\right)^{-1}, \qquad d_{n_0} > q\left(\ln \frac{sd_1}{qc_1}\right)^{-1}.$$

Then for $S_{m,n}$ *defined by* (5) *we have*

(20) $$\|S_{m,n}(1/v_{p,q}(t,z); \cdot, \cdot)\|_{r,s} \leq 1 \quad \textit{for} \quad m > m_0,\ n > n_0.$$

Moreover, for every $f \in C_{p,q}$ *and* $m > m_0,\ n > n_0$,

$$\|S_{m,n}(f)\|_{r,s} \le \|f\|_{p,q}. \tag{21}$$

From (5) *and* (21) *we deduce that* $S_{m,n}$, $m > m_0,\ n > n_0$, *is a positive linear operator from the space* $C_{p,q}$ *into* $C_{r,s}$.

PROOF. From (1), (2), (5) and (9) it follows that

$$\begin{aligned} &v_{r,s}(x,y)S_{m,n}(1/v_{p,q}(t,z);a_m,b_m,c_n,d_n;x,y) = \\ &= \{v_r(x)S_m(1/v_p(t);a_m,b_m;x)\}\{v_s(y)S_n(1/v_q(z);c_n,d_n;y)\}, \end{aligned}$$

for all $(x,y) \in R_0^2$ and $m,n \in N$, which by (18) and (3) implies (20).

Moreover from (5) and (1) – (3) we get

$$\|S_{m,n}(f)\|_{r,s} \le \|f\|_{p,q}\|S_{m,n}(1/v_{p,q})\|_{r,s}, \qquad m > m_0,\ n > n_0$$

and next by (20) follows (21).

3. THEOREMS

3.1. First we shall give theorems on the degree of approximation of $f \in C_{p,q}$ by $S_{m,n}(f)$ defined by (5). The partial derivatives of f we shall denote by f'_x, f'_y, f''_{xy}.

THEOREM 1. *Suppose that* $f \in C^1_{p,q}$ *with given* $p,q > 0$ *and* r, s, m_0, n_0 are *numbers satisfying the conditions of Lemma 4. Then there exists a positive constant* $M_2 \equiv M_2(p,q,r,s,a_1,b_1,c_1,d_1,m_0,n_0)$ *such that*

$$v_{r,s}(x,y)|S_{m,n}(f;x,y) - f(x,y)| \le M_2\left\{\|f'_x\|_{p,q}\sqrt{\frac{x}{b_n}} + \|f'_y\|_{p,q}\sqrt{\frac{y}{d_n}}\right\}, \tag{22}$$

for all $(x,y) \in R_0^2$ *and* $m > m_0,\ n > n_0$.

PROOF. Choosing $(x,y) \in R_0^2$ we can write for $f \in C^1_{p,q}$

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)\,du + \int_y^z f'_v(x,v)\,dv, \qquad (t,z) \in R_0^2.$$

From this and by (5) and (8) we get

$$(23)\quad S_{m,n}(f(t,z);x,y)-f(x,y)= S_{m,n}\left(\int_x^t f_u'(u,z)du;x,y\right)+$$

$$+S_{m,n}\left(\int_y^z f_v'(x,v)dv;x,y\right):=A_1+A_2 \qquad \text{for} \quad m,n\in N.$$

By (1) – (3) and (5) – (10) and next by the Hölder inequality and (18) we get

$$v_{r,s}(x,y)|A_1|\le v_{r,s}(x,y)S_{m,n}\left(\left|\int_x^t f_u'(u,z)du\right|;x,y\right)\le$$

$$\le \| f_x'\|_{p,q}\, v_{r,s}(x,y)S_{m,n}\left(\left(\frac{1}{v_{p,q}(t,z)}+\frac{1}{v_{p,q}(x,z)}\right)|t-x|;x,y\right)=$$

$$=\| f_x'\|_{p,q}\, v_s(y)S_n\left(\frac{1}{v_q(z)};y\right)\left\{v_r(x)S_m\left(\frac{|t-x|}{v_p(t)};x\right)+\frac{v_r(x)}{v_p(x)}S_m(|t-x|;x)\right\}\le$$

$$\le \| f_x'\|_{p,q}\left\{[v_r(x)S_m((t-x)^2/v_p(t);x)]^{1/2}[v_r(x)S_m(1/v_p(t);x)]^{1/2}+\right.$$

$$\left.+\frac{v_r(x)}{v_p(x)}[S_m((t-x)^2;x)]^{1/2}\right\}, \qquad m>m_0,\ n>n_0.$$

Applying (11) and Lemma 2, we deduce that

$$v_{r,s}(x,y)|A_1|\le M_3\|f_x'\|_{p,q}\sqrt{\frac{x}{b_m}}, \qquad \text{for} \quad m>m_0,\ \ n>n_0,$$

where $M_3\equiv M_3(p,r,a_1,b_1,m_0)=\text{const}>0$. Analogously we obtain

$$v_{r,s}(x,y)|A_2|\le M_4\| f_y'\|_{p,q}\sqrt{\frac{y}{d_n}}, \qquad \text{for} \quad m>m_0,\ \ n>n_0,$$

where $M_4\equiv M_4(q,s,c_1,d_1,n_0)=\text{const}>0$. From this and (23) follows (22).

THEOREM 2. *Let* $f\in C_{p,q}$ *with fixed* $p,q>0$ *and let* r, s, m_0, n_0 *are given numbers satisfying the conditions of Lemma 4. Then there exists a positive constant* $M_5\equiv M_5(p,q,r,s,a_1,b_1,c_1,d_1)$ such that

$$v_{r,s}(x,y)|S_{m,n}(f;x,y)-f(x,y)|\le M_5\,\omega\left(f,C_{p,q};\sqrt{\frac{x}{b_m}},\sqrt{\frac{y}{d_n}}\right), \tag{24}$$

for all $(x,y)\in R_0^2$ and $m>m_0,\ n>n_0$.

PROOF. We shall use the Stieklov function $f_{h,\delta}$,

$$f_{h,\delta}(x,y):=\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv, \qquad (x,y)\in R_0^2, \quad h,\delta>0, \tag{25}$$

of $f\in C_{p,q}$. From (25) follows

$$(f_{h,\delta})'_x(x,y)=\frac{1}{h\delta}\int_0^\delta(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y))dv,$$

$$(f_{h,\delta})'_y(x,y)=\frac{1}{h\delta}\int_0^h(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y))du,$$

and next by (3) and (4) we get

$$\|f_{h,\delta}-f\|_{p,q}\le\omega(f,C_{p,q};h,\delta), \tag{26}$$

$$\|(f_{h,\delta})'_x\|_{p,q}\le 2h^{-1}\omega(f,C_{p,q};h,\delta), \tag{27}$$

$$\|(f_{h,\delta})'_y\|_{p,q}\le 2\delta^{-1}\omega(f,C_{p,q};h,\delta), \tag{28}$$

for $h,\delta>0$, which prove that $f_{h,\delta}\in C^1_{p,q}$ if $f\in C_{p,q}$. Hence for $S_{m,n}$ defined by (5) we can write

$$v_{r,s}(x,y)|S_{m,n}(f;x,y)-f(x,y)|\le v_{r,s}(x,y)\left\{|S_{m,n}(f-f_{h,\delta};x,y)|+\right.$$

$$\left.+|S_{m,n}(f_{h,\delta};x,y)-f_{h,\delta}(x,y)|+|f_{h,\delta}(x,y)-f(x,y)|\right\}:=B_1+B_2+B_3,$$

for $(x,y)\in R_0^2$, $m,n\in N$ and $h,\delta>0$. By (1) – (3), (21) and (26) we get

$$B_3\le\omega(f,C_{p,q};h,\delta),$$

$$B_1\le\|S_{m,n}(f-f_{h,\delta};\cdot,\cdot)\|_{r,s}\le\|f-f_{h,\delta}\|_{p,q}\le\omega(f,C_{p,q};h,\delta),$$

for all $m>m_0$ and $n>n_0$. Applying Theorem 1 and (27) and (28), we obtain

$$B_2\le M_2\left\{\|(f_{h,\delta})'_x\|_{p,q}\sqrt{\frac{x}{b_m}}+\|(f_{h,\delta})'_y\|_{p,q}\sqrt{\frac{y}{d_n}}\right\}\le$$

$$\leq 2M_2\omega(f,C_{p,q};h,\delta)\left\{h^{-1}\sqrt{\frac{x}{b_m}}+\delta^{-1}\sqrt{\frac{y}{d_n}}\right\}, \qquad m>m_0,\ n>n_0.$$

Consequently

(29) $$v_{r,s}(x,y)|S_{m,n}(f;x,y)-f(x,y)|\leq$$

$$\leq 2(1+M_2)\omega(f,C_{p,q};h,\delta)\left\{1+h^{-1}\sqrt{\frac{x}{b_m}}+\delta^{-1}\sqrt{\frac{y}{d_n}}\right\},$$

for all $(x,y)\in R_0^2$, $m>m_0$, $n>n_0$ and $h,\delta>0$. Next, for fixed $x,y>0$ and m, n, setting $h=\sqrt{x/b_m}$ and $\delta=\sqrt{y/d_n}$ to (29), we obtain (24).

If $x=0$, $y>0$ or $x>0$, $y=0$, then (23) we obtain similarly as in [2]. If $x=0=y$, then by (5) we have $S_{m,n}(f;0,0)=f(0,0)$, for all $m,n\in N$.

Thus the proof is completed.

Theorem 2 and the properties of modulus of continuity imply the following

COROLLARY 1. *Let $f\in C_{p,q}$ with given $p,q>0$. Then for every $(x,y)\in R_0^2$ we have*

(30) $$\lim_{m,n\to\infty} S_{m,n}(f;x,y) = f(x,y).$$

Moreover, the convergence (30) *follows uniformly on every rectangle* $0\leq x\leq x_0$, $0\leq y\leq y_0$.

3.2. Now we shall prove the inequalities of the Bernstein type for $S_{m,n}$.

THEOREM 3. *Let the assumptions of Theorem 2 are satisfied. Then for first partial derivatives of $S_{m,n}(f)$ defined by* (5) *we have*

(31) $$\|(S_{m,n}(f))'_x\|_{r,s} \leq (1+e^{p/b_1})\|f\|_{p,q}\,a_m,$$

(32) $$\|(S_{m,n}(f))'_y\|_{r,s} \leq (1+e^{q/d_1})\|f\|_{p,q}\,c_n,$$

for all $m>m_0$ and $n>n_0$.

PROOF. The formulas (5) and (6) imply

$$(S_{m,n}(f))'_x(x,y)=a_m\{-S_{m,n}(f(t,z);x,y)+S_{m,n}(f(t+1/b_m,z);x,y)\},$$

for all $(x,y)\in R_0^2$ and $m,n\in N$. From this and by (20), (21) and (1) – (3) we get

$$\begin{aligned}\|(S_{m,n}(f))'_x\|_{r,s} &\le a_m\{\|f\|_{p,q}+\|S_{m,n}(f(t+1/b_m,z);\cdot,\cdot)\|_{r,s}\}\le\\ &\le a_m\|f\|_{p,q}\{1+\|S_{m,n}(1/v_{p,q}(t+1/b_m,z);\cdot,\cdot)\|_{r,s}\}\le\\ &\le a_m\|f\|_{p,q}\{1+e^{p/b_m}\|S_{m,n}(1/v_{p,q}(t,z);\cdot,\cdot)\|_{r,s}\}\le\\ &\le a_m\|f\|_{p,q}(1+e^{p/b_1})\qquad\text{for}\quad m>m_0,\ n>n_0.\end{aligned}$$

The proof of (32) is analogous.

3.3. In this section we shall consider the operators $S_{n,n}(f;\cdot,\cdot)\equiv$ $\equiv S_{n,n}(f;a_n,b_n,c_n,d_n;\cdot,\cdot)$, $n\in N$, $f\in C_{p,q}$, where the sequences (a_n), (b_n), (c_n) and (d_n) satisfy conditions of definition (5) and moreover

(33) $$\frac{b_n}{d_n}=1+o(1)\qquad\text{as}\quad n\to\infty.$$

For $S_{n,n}$ we shall prove the Voronovskaya type theorem and theorem on the convergence of first partial derivatives. Let $R_+^2=\{(x,y):x>0,y>0\}$.

THEOREM 4. *Suppose that $f\in C_{p,q}^2$ with given $p,q>0$. Then for every $(x,y)\in R_+^2$ we have*

(34) $$\lim_{n\to\infty}b_n\{S_{n,n}(f;x,y)-f(x,y)\}=\frac{x}{2}f''_{xx}(x,y)+\frac{y}{2}f''_{yy}(x,y).$$

PROOF. Let $(x,y)\in R_+^2$ be fixed point. Then by the Taylor formula for $f\in C_{p,q}^2$ we have

$$\begin{aligned}f(t,z)&=f(x,y)+f'_x(x,y)(t-x)+f'_y(x,y)(z-y)+\\ &+\tfrac{1}{2}(f''_{xx}(x,y)(t-x)^2+2f''_{xy}(x,y)(t-x)(z-y)+f''_{yy}(x,y)(z-y)^2)+\\ &+\varepsilon_1(t,z;x,y)\sqrt{(t-x)^4+(z-y)^4},\qquad(t,z)\in R_0^2,\end{aligned}$$

where $\varepsilon_1(t,z)=\varepsilon_1(t,z;x,y)$ is a function belonging to $C_{p,q}$ and $\varepsilon_1(x,y)=0$. From this and by (5) – (10) it follows that

$$S_{n,n}(f(t,z);x,y)=f(x,y)+f'_x(x,y)S_n(t-x;a_n,b_n;x)+$$
$$+f'_y(x,y)S_n(z-y;c_n,d_n;y)+\tfrac{1}{2}\{f''_{xx}(x,y)S_n((t-x)^2;a_n,b_n;x)+$$
$$+2f''_{xy}(x,y)S_n(t-x;a_n,b_n;x)S_n(z-y;c_n,d_n;y)+$$
$$+f''_{yy}(x,y)S_n((z-y)^2;c_n,d_n;y)\}+T_n,\qquad n\in N,$$

where

(35) $$T_n:=S_{n,n}(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y).$$

Applying (11), Lemma 3 and (33), we get

(36) $$\lim_{n\to\infty} b_n\,\{S_{n,n}(f;x,y)-f(x,y)\}=\frac{x}{2}f''_{xx}(x,y)+\frac{y}{2}f''_{yy}(x,y)+\lim_{n\to\infty}(b_nT_n).$$

By the Hölder inequality and (5) – (10) we get from (35)

(37) $$|T_n|\le\{S_{n,n}(\varepsilon_1^2(t,z);x,y)\}^{1/2}\{S_n((t-x)^4;a_n,b_n;x)+$$
$$+S_n((z-y)^4;c_n,d_n;y)\}^{1/2}.$$

The properties of ε_1 and Corollary 1 imply

(38) $$\lim_{n\to\infty} S_{n,n}(\varepsilon_1^2(t,z);x,y)\;=\;\varepsilon_1^2(x,y)\;=\;0.$$

Applying (38) and Lemma 3, we derive from (37)

$$\lim_{n\to\infty}(b_n\,T_n)\;=\;0.$$

From this and (36) follows (34).

THEOREM 5. *Suppose that* $f\in C^1_{p,q}$ *with given* $p,q>0$. *Then for every* $(x,y)\in R^2_+$ *we have*

(39) $$\lim_{n\to\infty}(S_{n,n}(f))'_x(x,y)\;=\;f'_x(x,y),$$

(40) $$\lim_{n\to\infty}(S_{n,n}(f))'_y(x,y)\;=\;f'_y(x,y).$$

PROOF. We shall prove only (39) because the proof of (40) is analogous. From (5) and (6) we deduce that

$$(S_{n,n}(f))'_x(x,y) = -a_n S_{n,n}(f(t,z);x,y) + \frac{b_n}{x} S_{n,n}(tf(t,z);x,y) =$$

$$= (b_n - a_n) S_{n,n}(f(t,z);x,y) + \frac{b_n}{x} S_{n,n}((t-x)f(t,z);x,y),$$

$(x,y) \in R_+^2$, $n \in N$. By the Taylor formula for $f \in C_{p,q}^1$ and fixed $(x,y) \in R_+^2$ we have

$$f(t,z) = f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(z-y) + \lambda(t,z;x,y), \qquad (t,z) \in R_0^2,$$

where

(41) $$\lambda(t,z;x,y) := \varepsilon_2(t,z;x,y)\sqrt{(t-x)^2 + (z-y)^2},$$

and $\varepsilon_2(t,z) \equiv \varepsilon_2(t,z;x,y)$ is a function belonging to $C_{p,q}$ and $\varepsilon_2(x,y) = 0$. From the above and by (5) – (10) we derive

(42) $$(S_{n,n}(f))'_x(x,y) = (b_n - a_n)\{f(x,y) + f'_x(x,y) S_n(t-x;a_n,b_n;x) +$$
$$+ f'_y(x,y) S_n(z-y;c_n,d_n;y) + S_{n,n}(\lambda(t,z;x,y);x,y)\} +$$
$$+ \frac{b_n}{x}\{f(x,y) S_n(t-x;a_n,b_n;x) + f'_x(x,y) S_n((t-x)^2;a_n,b_n;x) +$$
$$+ f'_y(x,y) S_n(t-x;a_n,b_n;x) S_n(z-y;c_n,d_n;y) +$$
$$+ S_{n,n}((t-x)\lambda(t,z;x,y);x,y)\},$$

for $n \in N$. The properties of ε_2 and Corollary 1 and (41) imply

(43) $$\lim_{n\to\infty} S_{n,n}(\lambda(t,z;x,y);x,y) = \lambda(x,y;x,y) = 0,$$

(44) $$\lim_{n\to\infty} S_{n,n}(\varepsilon_2^2(t,z);x,y) = \varepsilon_2^2(x,y) = 0.$$

Arguing as in the proof of Theorem 4, we get

$$|S_{n,n}((t-x)\lambda(t,z;x,y);x,y)| \le \{S_{n,n}(\varepsilon_2^2(t,z);x,y)\}^{1/2}\{S_n((t-x)^4;a_n,b_n;x) +$$
$$+ S_n((t-x)^2;a_n,b_n;x) S_n((z-y)^2;c_n,d_n;y)\}^{1/2},$$

which by (44) and Lemma 3 yields

$$\lim_{n\to\infty} b_n S_{n,n}((t-x)\lambda(t,z;x,y);x,y)=0 \tag{45}$$

Applying (43), (45), Lemma 3 and the properties (i), (ii) and (33), we immediately obtain (39) from (42).

Finaly, we observe that the above theorems extend some results (see [3]) obtained for the classical Szasz-Mirakyan operators, i. e. $S_{m,n}$ defined by (5) with $a_n = b_n = c_n = d_n = n$, $n \in N$.

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