FASCICULI MATHEMATICI

Nr 32 2001

JAN KOROŃSKI

ON THE HOMOGENEOUS QUASI POLYHARMONIC POLYNOMIALS

ABSTRACT: In this paper the effective construction of maximal system linearly independent homogeneous quasi polyharmonic polynomials of degree n of q independent variables is given.

KEY WORDS: polyharmonic equations, quasi polyharmonic polynomials, linear independence.

1. In the present paper we shall give the construction of the system of homogeneous quasi polyharmonic polynomials of degree n and depending on q independent variables of the from

$$P_n^q(x_1, x_2, ..., x_q) = \sum_{p_1+p_2+...+p_q=n} a_{p_1 p_2 ... p_q} x_1^{p_1} x_2^{p_2} ... x_q^{p_q}, \tag{1}$$

where p_i, $i = 1,2,...,q$, are nonegative integers. The polynomials (1) satatisfy, the quasi p-harmonic equation

$$L^p u = 0, \quad p \in N, \tag{2}$$

where

$$Lu = \sum_{i,j=1}^{q} a_{ij} D^2_{x_i x_j} u, \qquad L^p = L(L^{p-1}), \tag{3}$$

and

$$A = [a_{ij}], \quad a_{ij} \in R, \quad i,j = 1,2,...,q \tag{4}$$

is the positive definite matrix.

In the monograph [1] the similar problem for $p=1$, $q=3$ and for the diagonal matrix A is solved. The problem is considered by the complicated method by curvilignes coordinates defined by the family of convenient quadrics.

In the paper [4] the similar problem for the symmetric matrix A by another method is solved.

2. Let us consider the case in which $p=1$, $q \in N$, $n \in N$, and the equation

$$Lu = 0. \tag{3a}$$

Let us consider the positive definite quadratic from

(5) $$F(x)=\sum_{i,j=1}^{q} a_{ij}x_i x_j.$$

Let

(6) $$C=[C_{ij}], \quad i,j=1,2,\ldots,q,$$

denote a nonsingular matrix such that the transformation

(T) $$x_i=\sum_{j=1}^{q} C_{ij}t_j, \qquad i=1,\ldots,q$$

transforms the quadratic from (5) to the canonical from

(7) $$\sum_{i=1}^{q} t_i^2.$$

Let

(T^{-1}) $$t_i=\sum_{j=1}^{q} k_{ij}x_j, \qquad i=1,\ldots,q$$

denote the inverse transformation to the transformation (T). It is known [2], that there exist exactly

$$h(n,q,1)=\binom{n+q-1}{q-1}-\binom{n+q-3}{q-1}$$

linearly independent harmonic homogeneous polynomials of degree n of q independent variables $t_1,..,t_q$ of the form

(8) $$W_i^{n,1}(t)=\sum_{p_1+p_2+\ldots+p_q=n} b_n^{p_1p_2\ldots p_q}t_1^{p_1}t_2^{p_2}\ldots t_q^{p_q}, \quad i=1,\ldots,h(n,q,1),$$

where p_i, $i=1,\ldots,q$, are nonegative integers and $t=(t_1,\ldots,t_q)$.

Let $x=(x_1,\ldots,x_q)$ and let

$$r^2(x)=\sum_{i=1}^{q}\left(\sum_{j=1}^{q} k_{ij}x_j\right)^2, \quad R^2(t)=\sum_{i=1}^{q} t_i^2$$

3. Now we shall prove the following

LEMMA 1. *There exist exactly* $h(n,q,1)$ *linearly independent quasi harmonic homogeneous polynomials* $Q_i^{n,1}(x)$, $i=1,\ldots,h(n,q,1)$ *satifying equation* (3a).

PROOF. Let M denotes the number of the polynomials $Q_i^{n,1}(x)$ linearly independent. Let us assume that $M < h(n,q,1)$. It is known [3] that the polynomials $W_j^{n,1}(t)$, $j = 1,2,...,M$ satisfty the equation (3a) with respect to variable t, and it is known that the condition

(9) $$\sum_{i=1}^{h(n,q,1)} C_i W_j^{n,1}(t) = 0$$

implies the conditions $C_i = 0$, $i = 1,...,h(n,q,1)$. Let us suppose, that by identity

$$\sum_{j=1}^{M} m_j Q_j^{n,1}(t) = 0$$

imply that $m_j = 0$, $j = 1,...,M$.

By the transformation (T^{-1}) we get

$$\sum_{j=1}^{M} m_j W_j^{n,1}(t) = 0$$

and $m_j = 0$, $j = 1,2,...,M$ what is contradics with the conditions (9). Similarly for $M > h(n,q,1)$ we obtain the contradiction. Therefore $M = h(n,q,1)$.

4. Let us consider the system of the polynomials

(10_1) $$W_i^{n,1}(t) = H_i^n(t), \quad i = 1,...,h(n,q,1),$$

where $H_i^n(t)$, $i = 1,2,...,h(n,q,1)$, are harmonic linearly independent homogeneous polynomials of the degree n of q variables, t_i, $i = 1,2,...,q$.

(10_2) $$W_i^{n,2}(t) = R^2(t) H_i^{n-2}(t), \quad i = 1,2,...,h(n-2,q,1),$$

where $H_i^{n-2}(t)$, $i = 1,2,...,h(n-2,q,1)$, are the harmonic homogeneous polynomials of the degree $n-2$ linearly independent of q variables t_i, $i = 1,2,...,q$, and so one

....

(10_p) $$W_i^{n,p}(t) = R^{2p-2}(t) H_i^{n-2p+2}(t), \quad i = 1,2,...,h(n-2p+2,q,1),$$

where $H^{n-2p+2}(t)$, $i = 1,2,...,h(n-2p+2,q,1)$, are the harmonic homogeneous polynomials of the degree $n-2p+2$ linearly independent, of the q variables t_i, $i = 1,...,q$.

The number of all the polynomials of the system (10_i), $i=1,...,p$ is equal $h(n,q,1)+h(n-2,q,1)+...+h(n-2p+2,q,1)=h(n,q,p)$.

By the paper [2] we shall give the following.

LEMMA 2. *The system of the polynomials $W_i^{n,p}(t)$, $i=1,...,p$, is the system of the linearly independent polynomials.*

PROOF. Let

$$W(t)=\sum_{i=1}^{p} W_i(t) \tag{11}$$

where

$$W_1(t)=\sum_{i=1}^{h(n,q,1)} C_i^{(1)} W_i^{n,1}(t),$$

$$W_2(t)=\sum_{i=1}^{h(n,q,1)} C_i^{(2)} W_i^{n,2}(t),$$

and so one

....

$$W_p(t)=\sum_{i=1}^{h(n-2p+2,q,1)} C_i^{(p)} W_i^{n,p}(t),$$

where $C_i^{(j)}$ are the constants.

By the [3], p. 196 and by (10_j), $j=1,2,...,p$, we obtain

$$\Delta^p W(t)=0 \tag{12}$$

and

$$\Delta^{p-1}W(t)=\Delta^{p-1}W_p(t)=A(n,q,p)\sum_{i=1}^{h(n-2p+2,q,1)} C_i^{(p)} H_i^{n-2p+2}(t)=0,$$

where $A(n,q,p)$ is a constant. By the linearly independent of the polynomials H_i^{n-2p+2} we obtain the following conditions

$$C_i^{(p)}=0, \qquad i=1,...,h(n-2p+2,q,1). \tag{13}$$

Similarly we obtain

$$\Delta^{p-2}W(t)=\Delta^{p-2}W_{p-1}(t)=A_1(n,q,p)\sum_{i=1}^{h(n-2p+4,q,1)} C_i^{(p-1)} H_i^{n-2p+4}(t)=0.$$

From above conditions we have

(14) $$C_i^{(n-1)} = 0, \quad i = 1,2,\dots,h(n-2p+4,q,1).$$

In and we obtain similarly the equations

$$C_i^{(1)} = 0, \qquad i = 1,2,\dots,h(n,q,1).$$

which completes the proof of Lemma 2.

5. Now we shall give the system $h(n,q,p)$ of the quasi p-harmonic homogeneous polynomials $Q_i^{n,p}(x)$ of degree n. Let us consider the system of the quasi p-harmonic polynomials

(15) $$\begin{cases} Q_i^{n,1}(x), \quad i = 1,\dots,h(n,q,1), \\ Q_i^{n,2}(x) = r^2(x)Q^{n-2,1}(x), \quad i = 1,\dots,h(n-2,q,1), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Q_i^{n,p}(x) = r^{2p-2}(x)Q^{n-2p+2,1}(x), \quad i = 1,\dots,h(n-2p+2,q,1), \end{cases}$$

By the lemmas 1, 2, we obtain following

THEOREM. *The system of the polynomials* (15) *which number is equal* $h(n,q,p)$ *is of maximal system linearly independent homogeneous quasi p-harmonic polynomials of the degree n, of the q variables* $x_1,\dots,x_q$.

REFERENCES

[1] E. Hobson, *The theory of spherical and ellipsoidal harmonics,* Cambridge, 1955.

[2] S. Tarnowski, On certain problem concerning the polyharmonic polynomials, *Demonstratio Math.* Vol. XIV, 2(1981)

[3] M. Krzyżański, Partial differential equations od second order, Vol. I, Warszawa, 1972.

[4] J. Musiałek, *On homogeneous polyharmonic polynomials*, Comm. Math. XI, 1968.

[5] H. Niemayer, Lokale und asymptotiche Eigenschaften der Losungen der Helmholtzschen Schwingungsgleichung, Jahresbericht der Deutschen Mathematiker Vereinigung, 65. Band. Heft I, ausgegeben am. 23.8.1962, Stutgart, 1-44.

(Cracow University of Technology, Faculty of Applied Physics and Computer Modelling, Institute of Mathematics, Kraków)

Received on 09.08.2000 and, in revised form, on 16.03.2001.