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EXISTENCE OF NONOSCILLATORY SOLUTION FOR LINEAR NEUTRAL DELAY EQUATION

ABSTRACT: Consider the neutral delay differential equation with positive and negative coefficients

$$\frac{d}{dt}[x(t)+px(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0$$

where $p\in R$ and

$$\tau\in(0,\infty),\quad \sigma_1,\sigma_2\in[0,\infty]\quad \text{and}\quad Q_1,Q_2\in C[[t_0,\infty),R^+].$$

We obtain the sufficient condition for the existence of a nonoscillatory solution of the above equation to be $\int^\infty Q_i(s)ds<\infty,\ \ i=1,2$ and certain technical conditions implying that $\int_t^\infty Q_1(s)ds$ dominates $\int_t^\infty Q_2(s)ds$ for t large enough, for $p\neq -1$.

KEY WORDS: nonoscillation, asymptotic behavior.

1. INTRODUCTION

Consider the neutral delay differential equation with positive and negative coefficients

$$\frac{d}{dt}[x(t)+px(t-\tau)]+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0 \tag{1}$$

where $p\in R$ and

$$\tau\in(0,\infty),\quad \sigma_1,\sigma_2\in[0,\infty]\quad \text{and}\quad Q_1,Q_2\in C[[t_0,\infty)R^+]. \tag{2}$$

Recently, there has been some activity concerning the oscillation and asymptotic behavior of Eq(1), see [2], [5], [6], [7], [8] and [9] directed mainly on the so-called linearized oscillation theory, see [3], [4] and [1] for the review and applications of this theory. This theory for Equation 1 was restricted to the case $p\in(0,1)$ and in the case where $Q_2(t)\equiv 0$, was restricted to the case $p>-1$, see [8]. The case $Q_2(t)\equiv 0$ was investigated in [11] where it was shown that the condition

$$\int^\infty Q_1(s)ds<\infty,$$

implies the existence of a nonoscillatory solution of Eq(1) for $p\neq -1$. As is known if $p=-1$, $Q_2(t)\equiv 0$ and

$$\int^{\infty} Q_1(s)ds = \infty,$$

then all solutions of Eq(1) oscillate, see [10]. Even in the case where the above integral is finite and $Q_2(t) \equiv 0$ one can have the oscillation of all solutions for $p=-1$, see [12]. In general all above mentioned results hold only for some values of parametr p. Here we obtain the global (with respect to p) result in the non-constant coefficient case. Our result is the sufficient condition for the existence of a nonoscillatory solution for all values of $p \neq -1$.

Let $m = \max\{\tau, \sigma_1, \sigma_2\}$. By solution of Equation (1) we mean a function $y \in C[[t_1 - m, \infty), R]$, for some $t_1 \geq t_0$, such that $y(t) + py(t-\tau)$ is continuously differentiable on $[t_1, \infty)$ and such that Equation (1) is satisfied for $t \geq t_1$.

Assume that (2) holds, $t_1 \geq t_0$ and let $\phi \in C[[t_1 - m, t_1], R]$ be a given initial function. Then one can easily see by the method of steps that Equation (1) has a unique solution $y \in C[[t_1 - m, \infty), R]$ such that

$$y(t) = \phi(t) \quad \text{for} \quad t_1 - m \leq t \leq t_1.$$

As is customary, a solution of Equation (1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

2. Main result

Theorem 1. *Consider Equation* (1) *subject to the conditions* (2),

$$\int^{\infty} Q_i(s)ds < \infty, \quad \textit{for} \quad i = 1,2, \tag{3}$$

$$\int_t^{\infty} (aQ_1(s) - Q_2(s))ds \leq \int_{T_1}^{\infty} (aQ_1(s) - Q_2(s))ds,$$

$$\int_t^{\infty} \left(\frac{1}{a} Q_1(s) - Q_2(s) \right) ds > 0 \quad \textit{for enery} \quad t \geq T_1 \ \textit{and} \ a > 1 \tag{4}$$

where $p \neq -1$ *and* T_1 *is large enough. Then Equation* (1) *has a nonoscillatory solution.*

Proof. The proof of this theorem will be divided into five claims depending on the five different ranges of the parametr p.

CLAIM 1. $p \in [0,1)$

Choose a t_1 sufficeintly large such that

$$t_1 \geq \max\{T_1, t_0 + \sigma\}, \qquad \sigma = \max\{\tau, \sigma_1, \sigma_2\}, \tag{5}$$

and

$$\int_{t_1}^{\infty} [Q_1(s) + Q_2(s)]ds < 1 - p, \tag{6}$$

$$\int_{t}^{\infty} [M_2 Q_1(s) - M_1 Q_2(s)]ds \leq \int_{t_1}^{\infty} [M_2 Q_1(s) - M_1 Q_2(s)]ds \leq \leq p - 1 + M, \quad t > t_1, \tag{7}$$

$$\int_{t}^{\infty} [M_1 Q_1(s) - M_2 Q_2(s)]ds \geq 0, \quad \text{for} \quad t \geq t_1 \tag{8}$$

hold, where M_1 and M_2 are positive constans such that

$$\frac{1 - M_1}{1 + M_2} \geq p > 1 - M_2.$$

Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup-norm. Set

$$A = \{x \in X : M_1 \leq x(t) \leq M_2,\ t \geq t_0\}.$$

Define a mapping $T : A \to X$ as follows

$$(Tx)(t) = \begin{cases} 1 - p - px(t-\tau) + \int_{t}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds, & t \geq t_1, \\ (Tx)(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

Clearly Tx is continuous. For every $x \in A$ and $t \geq t_1$ using (7) we get

$$\begin{aligned}(Tx)(t) &= 1 - p - px(t-\tau) + \int_{t}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \leq \\ &\leq 1 - p + \int_{t}^{\infty} [M_2 Q_1(s) - M_1 Q_2(s)]ds \leq \\ &\leq 1 - p + \int_{t_1}^{\infty} [M_2 Q_1(s) - M_1 Q_2(s)]ds \leq \\ &\leq 1 - p + p - 1 + M_2 = M_2.\end{aligned}$$

Further, in view of (8) have

$$(Tx)(t) = 1 - p - px(t-\tau) + \int_t^\infty [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \le$$

$$\le 1 - p - pM_2 + \int_t^\infty [M_1Q_1(s) - M_2Q_2(s)]ds \le$$

$$\le 1 - p - pM_2 \ge M_1.$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed and convex subset of X we have to prove that T is a contraction mapping on A in order to apply the contraction principle.

Now, for $x_1, x_2 \in A$ and $t \ge t_1$ we have

$$|(Tx_1)(t) - (Tx_2)(t)| \le p|x_1(t-\tau) - x_2(t-\tau)| +$$

$$+ \int_t^\infty [Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)| +$$

$$+ Q_2(s)|x_1(s-\sigma_2) - x_2(s-\sigma_1)|]ds \le$$

$$\le p\|x_1 - x_2\| + \|x_1 - x_2\| \int_t^\infty [Q_1(s) + Q_2(s)]ds =$$

$$= q_1\|x_1 - x_2\|,$$

where we use sup norm. This immediately implies that

$$\|Tx_1 - Tx_2\| \le q_1\|x_1 - x_2\|$$

where in view of (6) $q_1 < 1$ which proves that T is a contraction mapping. Consequently T has the fixed point x which is a positive solution of Equation (1) which completes the proof of Claim 1.

CLAIM 2. $p \in (1, \infty)$

Choose a $t_1 > T_1 > t_0$ sufficiently large such taht

(9) $$t_1 + \tau \ge t_0 + \max\{\sigma_1, \sigma_2\}$$

(10) $$\int_{t_1}^\infty [Q_1(s) + Q_2(s)]ds < p - 1,$$

(11) $$\int_t^\infty [N_2Q_1(s) - N_1Q_2(s)]ds \le \int_{t_1}^\infty [N_2Q_1(s) - N_1Q_2(s)]ds \le$$
$$\le 1 - p + pN_2, \quad t > t_1,$$

and

$$(12) \qquad \int_{t+\tau}^{\infty}[N_1Q_1(s)-N_2Q_2(s)]ds \ge 0, \quad \text{for} \quad t\ge t_1,$$

where N_1 and N_2 are positive constants such that

$$p \ge \frac{1+N_2}{1-N_1} \ge 0 \quad \text{and} \quad p(1-N_2) < 1.$$

Let X be the set as in Claim 1. Set

$$A = \{x\in X : N_1 \le x(t) \le N_2,\ t\ge t_0\}.$$

Define a mapping $T: A\to X$ as follows

$$(Tx)(t) = \begin{cases} 1-\frac{1}{p}-\frac{1}{p}x(t+\tau)+\frac{1}{p}\int_{t+\tau}^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds, & t\ge t_1, \\ (Tx)(t_1), \quad t_0\le t\le t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x\in A$ and $t\ge t_1$ using (11) we get

$$\begin{aligned}(Tx)(t) &= 1-\tfrac{1}{p}-\tfrac{1}{p}x(t+\tau)+\tfrac{1}{p}\int_{t+\tau}^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \le \\ &\le 1-\tfrac{1}{p}+\tfrac{1}{p}\int_{t+\tau}^{\infty}[N_2Q_1(s)-N_1Q_2(s)]ds \le \\ &\le 1-\tfrac{1}{p}+\tfrac{1}{p}\int_{t_1}^{\infty}[N_2Q_1(s)-N_1Q_2(s)]ds \le \\ &\le 1-\tfrac{1}{p}+\tfrac{1}{p}(1-p+pN_2) = N_2.\end{aligned}$$

Further in view of (12) we have

$$\begin{aligned}(Tx)(t) &= 1-\tfrac{1}{p}-\tfrac{1}{p}x(t+\tau)+\tfrac{1}{p}\int_{t+\tau}^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \ge \\ &\ge 1-\tfrac{1}{p}-\tfrac{N_2}{p}+\tfrac{1}{p}\int_{t+\tau}^{\infty}[N_1Q_1(s)-N_2Q_2(s)]ds \ge \\ &\ge 1-\tfrac{1}{p}-\tfrac{N_2}{p} \ge N_1.\end{aligned}$$

Thus we proved that $TA\subset A$. Since A is a bounded, closed and convex subset of X we have to prove that T is a contraction mapping on A in order to apply the contraction principle.

Now for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$|(Tx_1)(t) - (Tx_2)(t)| \leq \tfrac{1}{p}|x_1(t+\tau) - x_2(t+\tau)| +$$
$$+\tfrac{1}{p}\int_{t+\tau}^{\infty}[Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)| +$$
$$+ Q_2(s)|x_1(s-\sigma_2) - x_2(s-\sigma_1)|]ds \leq$$
$$\leq \tfrac{1}{p}\| x_1 - x_2 \| + \tfrac{1}{p}\| x_1 - x_2 \| \int_t^{\infty}[Q_1(s) + Q_2(s)]ds =$$
$$= q_2 \| x_1 - x_2 \|,$$

where we use sup norm. This immediately implies that

$$\| Tx_1 - Tx_2 \| \leq q_2 \| x_1 - x_2 \|$$

where in view of (10) $q_2 < 1$ which proves that T is a contraction mapping. Consequently T has the fixed point x which is a positive solution of Equation (1) which completes the proof of Claim 2.

CLAIM 3. $p = 1$

Choose a $t_1 > T_1 > t_0$ sufficiently large such that (9) holds and

$$\int_{t_1}^{\infty}[Q_1(s) + Q_2(s)]ds < 1, \tag{13}$$

$$0 \leq \sum_{i=1}^{\infty} \int_{t_1+(2i-1)\tau}^{t_1+2i\tau} [P_2 Q_1(s) - P_1 Q_2(s)]ds \leq P_2 - P, \tag{14}$$

$$\int_{t+(2i-1)\tau}^{t+2i\tau} [P_1 Q_1(s) - P_2 Q_2(s)]ds \geq 0, \quad t \geq t_1, \tag{15}$$

are satisfied, where the positive constants P_1 and P_2 satisfy $P_1 < P < P_2$.

Let X be the set as in Claim 1. Set

$$A = \{x \in X : P_1 \leq x(t) \leq P_2,\ t \geq t_0\}.$$

Define a mapping $T : A \to X$ as follows

$$(Tx)(t) = \begin{cases} P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$ using (14), we get

$$(Tx)(t) = P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \le$$

$$\le P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [P_2 Q_1(s) - P_1 Q_2(s)]ds \le$$

$$\le P + P_2 - P = P_2.$$

Further in view of (15) we have

$$(Tx)(t) = P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \ge$$

$$\ge P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [P_1 Q_1(s) - P_2 Q_2(s)]ds \ge$$

$$\ge P \ge P_1.$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed and convex subset of X we have to prove that T is a contraction mapping on A in order to apply the contraction principle.

Now for $x_1, x_2 \in A$ and $t \ge t_1$ we have

$$|(Tx_1)(t) - (Tx_2)(t)| \le \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s)|x(s-\sigma_1) - x(s-\sigma_2)| +$$

$$+ Q_2(s)|x_1(s-\sigma_2) - x_2(s-\sigma_2)|]ds \le$$

$$\le \|x_1 - x_2\| \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s) + Q_2(s)]ds \le$$

$$\le \|x_1 - x_2\| \sum_{i=1}^{\infty} \int_{t_1}^{t+2i\tau} [Q_1(s) + Q_2(s)]ds =$$

$$= q_3 \|x_1 - x_2\|,$$

where we use sup norm. This immediately implies that

$$\|Tx_1 - Tx_2\| \le q_3 \|x_1 - x_2\|$$

where in view of (13) $q_3 < 1$ which proves that T is a contraction mapping. Consequently T has the fixed point x, i.e.

$$x(t) = \begin{cases} P + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds, & t \ge t_1, \\ (Tx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

It follows that

$$x(t)+x(t-\tau) = 2P+\int_t^\infty [Q_1(s)x(s-\sigma_1)-Q_2(x)x(s-\sigma_1)]ds\,,\quad t\ge t_1+\tau.$$

From this we see that $x(t)$ is a positive solution of Equation (1) on $[t_1+\tau,\infty)$ and so the proof is complete.

CLAIM 4. $p\in(-1,0)$

Choose a $t_1>T_1>t_0$ sufficiently large such taht (5) and the following inequalities

(16) $$\int_{t_1}^\infty [Q_1(s)+Q_2(s)]ds < 1+p,$$

(17) $$0\le \int_t^\infty [M_4\,Q_1(s)-M_3Q_2(s)]ds \le \int_{t_1}^\infty [M_4\,Q_1(s)-M_3Q_2(s)]ds \le$$
$$\le (p+1)(M_4-1),$$

and every $t>t_1$ and

(18) $$\int_t^\infty [M_3\,Q_1(s)-M_4Q_2(s)]ds \ge 0,\qquad t\ge t_1,$$

hold, where the constants M_3 and M_4 satisfy:

$$0 < M_3 \le 1 < M_4.$$

Let X be the set as in Claim 1. Set

$$A = \{x\in X: M_3\le x(t)\le M_4,\ t\ge t_0\}.$$

Define a mapping $T:A\to X$ as follows

$$(Tx)(t)=\begin{cases} 1+p-p\,x(t-\tau)+\int_t^\infty [Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds, & t\ge t_1,\\ (Tx)(t_1), & t_0\le t\le t_1.\end{cases}$$

Clearly, Tx is continuous. For every $x\in A$ and $t\ge t_1$ using (17) we get

$$(Tx)(t) = 1+p-p\,x(t-\tau)+\int_t^\infty [Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \le$$

$$\le 1+p-pM_4+\int_t^\infty [M_4\,Q_1(s)-M_3Q_2(s)]ds \le$$

$$\leq 1+p-pM_4+\int_{t_1}^{\infty}[M_4Q_1(s)-M_3Q_2(s)]ds \leq$$

$$\leq 1+p-pM_4+(p+1)(M_4-1) = M_4.$$

Further, in view of (18) we have

$$(Tx)(t) = 1+p-px(t-\tau)+\int_t^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \geq$$

$$\geq 1+p-pM_3+\int_t^{\infty}[M_3Q_1(s)-M_4Q_2(s)]ds \geq$$

$$\geq 1-p-pM_3 \geq M_3.$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed and convex subset of X we have to prove that T is a contraction mapping.

Now for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$|(Tx_1)(t)-(Tx_2)(t)| \leq -p\,|x_1(t-\tau)-x_2(t-\tau)|+$$

$$+\int_t^{\infty}[Q_1(s)\,|x_1(s-\sigma_1)-x_2(s-\sigma_1)|+$$

$$+\ Q_2(s)\,|x_1(s-\sigma_2)-x_2(s-\sigma_1)|]ds \leq$$

$$\leq -p\,\|x_1-x_2\|+\|x_1-x_2\|\int_{t_1}^{\infty}[Q_1(s)+Q_2(s)]ds =$$

$$= q_4\,\|x_1-x_2\|,$$

where in view of (16) $q_4 < 1$. This proves that T is a contraction mapping. Consequently T has the fixed point x which is a positive solution of Equation (1) which completes the proof of Claim 4.

CLAIM 5. $p \in (-\infty, -1)$

Choose a $t_1 > t_0$ sufficiently large such that (9) and the following inequalities

(19) $$\int_{t_1}^{\infty}[Q_1(s)+Q_2(s)]ds < -1-p,$$

(20) $$0 \leq \int_t^{\infty}[N_4Q_1(s)-N_3Q_2(s)]ds \leq \int_{t_1}^{\infty}[N_4Q_1(s)-N_3Q_2(s)]ds \leq$$

$$\leq (p+1)(N_3-1), \qquad t > t_1$$

and

$$(21) \qquad \int_{t+\tau}^{\infty} [N_3 Q_1(s) - N_4 Q_2(s)]ds \geq 0 \quad \text{for} \quad t \geq t_1$$

hold, where the constants N_3 and N_4 satisfy:

$$0 < N_3 \leq 1 < N_4.$$

Let X be the set as in Claim 1. Set

$$A = \{x \in X : N_3 \leq x(t) \leq N_4,\ t \geq t_0\}.$$

Define a mapping $T : A \to X$ as follows

$$(Tx)(t) = \begin{cases} 1 + \frac{1}{p} - \frac{1}{p}x(t+\tau) + \frac{1}{p}\int_{t+\tau}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds, & t \geq t_1, \\ (Tx)(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$ using (21) we get

$$(Tx)(t) = 1 + \tfrac{1}{p} - \tfrac{1}{p}x(t+\tau) + \tfrac{1}{p}\int_{t+\tau}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \leq$$

$$\leq 1 + \tfrac{1}{p} - \tfrac{N_4}{p} + \tfrac{1}{p}\int_{t+\tau}^{\infty} [N_3 Q_1(s) - N_4 Q_2(s)]ds \leq$$

$$\leq 1 + \tfrac{1}{p} - \tfrac{N_4}{p} \leq N_4.$$

Further in view of (20) we have

$$(Tx)(t) = 1 + \tfrac{1}{p} - \tfrac{1}{p}x(t+\tau) + \tfrac{1}{p}\int_{t+\tau}^{\infty} [Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)]ds \geq$$

$$\geq 1 + \tfrac{1}{p} - \tfrac{N_3}{p} + \tfrac{1}{p}\int_{t+\tau}^{\infty} [N_4 Q_1(s) - N_3 Q_2(s)]ds \geq$$

$$\geq 1 + \tfrac{1}{p} - \tfrac{N_3}{p} + \tfrac{1}{p}(p+1)(N_3 - 1) = N_3.$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed and convex subset of X we have to prove that T is a contraction mapping on A in order to apply the contraction principle.

Now for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$|(Tx_1)(t) - (Tx_2)(t)| \leq -\tfrac{1}{p}|x_1(t+\tau) - x_2(t+\tau)| +$$

$$+\tfrac{1}{p}\int_{t+\tau}^{\infty}[Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_1)|+$$

$$+\,Q_2(s)|x_1(s-\sigma_2)-x_2(s-\sigma_1)|]ds\le$$

$$\le -\tfrac{1}{p}\|x_1-x_2\|-\tfrac{1}{p}\|x_1-x_2\|\int_{t}^{\infty}[Q_1(s)+Q_2(s)]ds=$$

$$= q_5\|x_1-x_2\|,$$

where we use sup norm. This immediately implies that

$$\|Tx_1-Tx_2\|\le q_5\|x_1-x_2\|$$

where in view of (19) $q_5<1$ which proves that T is a contraction mapping. Consequently T has the fixed point x which is a positive solution of Equation (1) which completes the proof of Claim 5.

The proof of the theorem is complete.

REMARK. The condition (4) which implies that $\int_t^\infty Q_1(s)ds$ dominates $\int_t^\infty Q_2(s)ds$ may look too restrictive. This condition is actually affected by the choice of the constants M_i, N_i and P_i, $i=1,2,3,4$. Choosing those constants in an appropriate way we can specify that this condition holds for a single value of a; in this case this condition becomes easy to check and use. For instance if $M_{2k}=\alpha M_{2k-1}$, $N_{2k}=\alpha N_{2k-1}$, $P_2=\alpha P_1$, $k=1,2$ then $a=\alpha$ in (4), where $\alpha>1$ is a given number. Choosing α to be as closed to 1 as we please, we get very precise asymptotic behavior for the nonoscillatory solution we constructed, since in all cases we have

$$M_{2k-1}\le x(t)\le \alpha M_{2k-1},\quad k=1,2$$

or

$$N_{2k-1}\le x(t)\le \alpha N_{2k-1},\quad k=1,2$$

or

$$P_1\le x(t)\le \alpha P_1.$$

We can also specify our choice of constants by choosing $M_1=M_3=N_1=N_3=P_1$ and $M_2=M_4=N_2=N_4=P_2$, which can be achieved by taking M_1 and M_2 to satisfy $0<M_1<M_2$ and $M_2^2>M_1$. In this case in all five cases we will have the same asymptotic behavior of nonoscillatory solution as $M_1\le x(t)\le M_2$ with the same value of $a=M_2/M_1$. Combining the last two choices of constants, we get $M_1\le x(t)\le \alpha M_1$ with $a=\alpha$.

Finally, in the special case where $Q_2(t) \equiv 0$ the condition (4) is redundant and theorem holds under condition (3) only. In this case we get the extension of the result in [11]. In this case the result seems to be sharp for $p > -1$ in the sense that if $\int^{\infty} Q_1(s)ds = \infty$, then either all solutions are oscillatory or

$$\liminf_{t\to\infty} y(t) = 0$$

for every nonoscillatory solution, see ([1], p. 18).

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