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PROXIMATE TYPE IN REFERENCE TO GENERALIZED BIAXISYMMETIC POTENTIALS

ABSTRACT: Let $F^{(\alpha,\beta)}(x,y)$ be a real valued regular solution to the generalized biaxially symmetric potential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial}{\partial x}+\frac{2\beta+1}{y}\frac{\partial}{\partial y}\right)F^{(\alpha,\beta)} = 0, \quad \alpha>\beta>-\frac{1}{2}.$$

To obtain a more refined measure of growth then is given by [1] an approximation theorem for arbitrary proximate types and some more asymptotic properties have been proved. The proximate type is constructed for a class of Generalized Biaxially Symmetric Potential (GBASP). Lastly, we obtain lower and upper bounds for proximate type in reference to growth parameters of GBASP.

KEY WORDS: generalized biaxially symmetric potentials, proximate type, Cauchy data, regular growth.

1. INTRODUCTION

Let $F^{(\alpha,\beta)} = F^{(\alpha,\beta)}(x,y)$ be a real valued regular solution to the generalized biaxially symmetric potential equation

$$(1.1)\qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{2\alpha+1}{x}\frac{\partial}{\partial x}+\frac{2\beta+1}{y}\frac{\partial}{\partial y}\right)F^{(\alpha,\beta)} = 0, \quad \alpha>\beta>-\frac{1}{2}.$$

subject to the Cauchy data

$$F_x^{(\alpha,\beta)}(0,y) = F_y^{(\alpha,\beta)}(x,0) = 0$$

along the singular lines in open hypersphere $\sum_R^{(\alpha,\beta)} : x^2+y^2<R^2$. These solutions, called the generalized biaxially symmetric potentials (GBASP) can be exponded in $\sum_R^{(\alpha,\beta)}$ uniquely as

$$(1.2)\qquad F^{(\alpha,\beta)}(x,y) = \sum_{n=0}^{\infty} a_n\, R_n^{(\alpha,\beta)}(x,y), \quad x,y\in R \ \text{(set of real numbers)}$$

in terms of the complete set

$$(1.3)\qquad R_n^{(\alpha,\beta)}(x,y) = P_n^{(\alpha,\beta)}[(x^2-y^2)/(x^2+y^2)]/P_n^{(\alpha,\beta)}(1),$$

of biaxisymmetric harmonic potentials, where $P_n^{(\alpha,\beta)}$ are Jacobi polynomials.

Now we consider a positive function $T(r)$ in $0<r<R,\ 0<R<\infty$, having the properties:

(i) $T(r)\to T$ as $r\to R,\ 0\le T\le\infty$;

(ii) $\dfrac{(R-r)T'(r)}{RT(r)}\to 0$ as $r\to R$

where $T'(r)$ denotes the derivatives of $T(r)$. Such a function $T(r)$ is called the proximate type.

2. Asymptotic Properties

Theorem 1. *For every continuously differentiable proximate type $T(r)$, there exists a twice continuously differentiable proximate type $S(r)$ such that*

$$(2.1)\qquad \lim_{r\to R}\frac{(R-r)^2\log((R-r)/R)S''(r)}{R^2S(r)} = 0,$$

and

$$(2.2)\qquad T(r)\cong S(r)\quad \textit{as}\quad r\to R.$$

Proof. Let us assume that $S(r)$ be a proximate type and coincide with $T(r)$ on the sequence $\{r_n\}$ in $[0,R)$ as

$$(2.3)\qquad T(r_n) = S(r_n),\qquad r_n = R\left(1-\frac{1}{4^n}\right),\quad n=0,1,2,\dots .$$

In this case, for r lying in the intervals $[r_n, r_{n+1})$

$$\log\frac{T(r)}{S(r)} = \left|\int_{r_n}^{r}\left\{\frac{T'(x)}{T(x)}-\frac{S'(x)}{S(x)}\right\}dx\right| = \left|\int_{r_n}^{r}o\left(\frac{R}{R-x}\right)dx\right| =$$

$$= o\left(\log\frac{R-r}{R-r_n}\right) = o(1)\quad\text{as}\quad r\to R.$$

Hence $T(r)=S(r)e^{o(1)}$, which implies (2.2).

Thus it is sufficient to construct a twice continuously differentiable proximate types $S(r)$ satisfying the conditions (2.1) and (2.3). Define the functions on the interval $[0,3/4]$:

$$\phi = \begin{cases} t & 0 \le t \le 1/4, \\ -2t + 3/4 & 1/4 \le t \le 1/2, \\ t - 3/4 & 1/2 \le t \le 3/4, \end{cases}$$

and

$$\xi(\sigma) = \int_0^{\sigma} \phi(t)\, dt .$$

Since $\phi(t)$ is continuous on $[0, 3/4]$, it follows that $\xi(\sigma)$ is continuously differentiable on $[0, 3/4]$. We also have

(a) $0 = \xi(0) = \xi(3/4) = \xi'(0) = \xi'(3/4)$,

(b) $0 \le \xi(\sigma) \le 3/16$,

(c) $|\xi'(\sigma)| \le 1/4$,

(d) $\int_0^{3/4} \xi(x)\, dx = \delta > 0$.

Define a sequence $\{\in_n\}$ such that

$$\in_n = \frac{\log (T(r_{n+1})/T(r_n))}{\delta} .$$

Since

$$\frac{T'(r)}{T(r)} = o\left(\frac{R}{R-r}\right),$$

it gives

$$\int_{r_n}^{r_{n+1}} \frac{T'(r)}{T(r)} dr = \int_{r_n}^{r_{n+1}} o\left(\frac{R}{R-r}\right) dr,$$

or

$$\log \frac{T(r_{n+1})}{T(r_n)} = o(\log 4).$$

Hence

$$\in_n \to 0 \quad \text{as} \quad n \to \infty \quad (\text{Taking } \delta = \log 4)$$

Finally, we define

(2.4) $$\log S(r) = \log T(r_n) + \frac{\in_n R}{R - r_n} \int_{r_n}^{r} \xi\left(\frac{t - r_n}{R - r_n}\right) dt$$

on the interval $[r_n, r_{n+1}]$.

The varification of properties (i) and (ii) and derivation of (2.1) for the positive function $S(r)$ in (2.4) can be easily obtain.

THEOREM 2. *Let $T(r)$ be a proximate type. Then, for ρ $(0<\rho<\infty)$ and T* $(0\le T<\infty)$,

(a) $\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}$ *is monotonically increasing for* $r>r_0$.

(b) $\exp\left\{\left(\frac{R-r-\rho}{R-r}\right)T(r+\rho)+\rho T\right\} \cong \exp\left\{\left(\frac{R-r}{R}\right)T(r)\right\}$ *as* $r\to R$.

PROOF. (a) We have

$$\frac{d}{dr}\left[\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}\right] = \frac{\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}}{(R/R-r)^{(\rho+1)}}\left[\rho T(r)+\left(\frac{R-r}{R}\right)T'(r)\right].$$

For $T>0$, the condition (ii) may be replaced by $((R-r)/R)T'(r)\to 0$ as $r\to R$. Thus, we get asymptotically

$$\frac{d}{dr}\left[\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}\right] > \frac{\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}}{(R/R-r)^{(\rho+1)}}(\rho T-\varepsilon), \quad (0<\varepsilon<\rho T).$$

In case, $T=0$, $(R/(R-r))^{\rho}T(r)\to\infty$ as $r\to R$ and

$$\frac{d}{dr}\left[\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}\right] = \frac{T(r)\exp\left\{\left(\frac{R}{R-r}\right)^{\rho} T(r)\right\}}{(R/R-r)^{-(\rho+1)}}\left(\rho+\frac{(R-r)T'(r)}{RT(r)})\right) >$$

$$> \frac{T(r)\exp\left\{\left(\frac{R}{R-r}\right)T(r)\right\}}{(R/R-r)^{-(\rho+1)}}(\rho-\varepsilon).$$

(b) For $T>0$, considering the function

(2.5) $$L(r) = \exp\{(R-r/R)(T(r)-T)\}.$$

Thus, for all values of $r \to R$,

$$\frac{L'(r)}{L(r)} = o(1)$$

or

$$\lim_{r \to R} \log \frac{L(r+\rho)}{L(r)} = 0. \tag{2.6}$$

In case, $T = 0$, (2.5) reduced to

$$\frac{L'(r)}{L(r)} = T(r)\left\{\frac{(R-r)T'(r)}{RT(r)} - 1\right\}.$$

In view of properties (i) and (ii), again $L'(r)/L(r) \to 0$ as $r \to R$ and (2.6) is available which means

$$L(r+\rho) \cong L(r) \quad \text{as } r \to R.$$

This immediatedy correspond to (b).

3. Construction and Bounds

Let a real valued GBASP $F^{(\alpha,\beta)}$ regular in $\sum_R^{(\alpha,\beta)}$ having order ρ $(0 < \rho < \infty)$, type T $(0 \le T \le \infty)$ and satisfying in addition (i) and (ii). Then for a given n $(0 < n < \infty)$, $T(r)$ satisfies also:

(iii) $T(r)$ is continuous and piecewise differentiable for $r > r_0$;

and

(iv) $\limsup_{r \to R} \dfrac{M(r, F^{(\alpha,\beta)})}{\exp\{(R/R-r)^{\rho}\, T(r)\}} = n, \; M(r, F^{(\alpha,\beta)}) = \max_{x^2+y^2=r^2} |F^{(\alpha,\beta)}(x,y)|.$

Now $T'(r)$ in (ii) can be interpreted as $T'(r^+)$ or $T'(r^-)$ whenever these are unequal and the comparison function $T(r)$ is called the proximate type of the given real valued GBASP. The existance of such comparison function established in [1]. Obviously, proximate type of a real valued GBASP is not uniquely determined. For example, if we add $\gamma/(R/R-r)^{\rho}$, $0 < \gamma < \infty$ in the proximate type $T(r)$ we, again obtain a new proximate type for the same GBASP and the corresponding value of n is divided by e^{γ}.

The GBASP are natural extensions of harmonic or analytic functions. Hence, we anticipate properties, similar to those of the harmonic functions found from associated analytic *f*, by taking Ref, the real part of *f*.

By the Hadamard three circle theorem, we know, if $f(z)$ is analytic in finite disc, $\log M(r,f)$ in an increasing convex function of $\log r$ in $0<r<R$. Using above theorem for $F^{(\alpha,\beta)}(x,y)$ we have if $F^{(\alpha,\beta)}(x,y)$ is regular in open hypersphere $\sum_R^{(\alpha,\beta)}$, $\log M(r,F^{(\alpha,\beta)})$ is an increasing convex function of $\log r$ in $0<r<R$.

Morever, it has the representation

$$\text{(3.1)}\quad \log M(r,F^{(\alpha,\beta)}) = \log M(r_0,F^{(\alpha,\beta)}) + \int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x}dx,\quad 0<r_0<r<R,$$

where $w(x,F^{(\alpha,\beta)})$ is a positive, continuous and piecewise differentiable function of x.

The existence of (3.1) established by using ([2], Lemma 1, eqs. 2.1, 2.2). Now we prove:

LEMMA 1. *For a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hypersphere $\sum_R^{(\alpha,\beta)}$ and having order ρ and lower order λ, we have*

$$\text{(3.2)}\quad \liminf_{r\to R} \frac{(R-r)\,w(r,F^{(\alpha,\beta)})}{Rr\log M(r,F^{(\alpha,\beta)})} \le \lambda < \rho \limsup_{r\to R} \frac{(R-r)\,w(r,F^{(\alpha,\beta)})}{Rr\log M(r,F^{(\alpha,\beta)})},$$

PROOF. Let R_+ be the set of extended positive real numbers. Then, for $A\in R_+\cup\{0\}$, we define

$$\text{(3.3)}\quad \limsup_{r\to R} \frac{w(r,F^{(\alpha,\beta)})}{r(R/R-r)} = A.$$

For $A=0$, $\rho=0$. On differentiation (3.1) gives

$$\text{(3.4)}\quad \frac{M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})} = \frac{w(r,F^{(\alpha,\beta)})}{r}.$$

The expression (3.3) together with (3.4) is rewritten as

$$\limsup_{r\to R} \frac{(R-r)\,M'(r,F^{(\alpha,\beta)})}{R\,M(r,F^{(\alpha,\beta)})\log M(r,F^{(\alpha,\beta)})} = A.$$

For given $\varepsilon>0$ and $r>r_0(\varepsilon)$,

$$\frac{M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})\log M(r,F^{(\alpha,\beta)})} < \frac{A+\in}{(R-r)/R}.$$

Integrating above inequality, we get

$$\log\log M(r, F^{(\alpha,\beta)}) < o(1) + (A+\varepsilon)\log(R/R-r).$$

Passing to limits, we get

$$\lambda \le \lim_{r\to R} \frac{(R-r)\,w(r, F^{(\alpha,\beta)})}{Rr\log M(r, F^{(\alpha,\beta)})},$$

which holds for $A=\infty$. Likewise, for lower order λ,

$$\lambda \ge \liminf_{r\to R} \frac{(R-r)\,w(r, F^{(\alpha,\beta)})}{Rr\log M(r, F^{(\alpha,\beta)})}.$$

Combining above two inequalities (3.2) is immediate.

LEMMA 2. *If a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hypersphere $\sum_R^{(\alpha,\beta)}$ having order ρ $(0<\rho<\infty)$, type T and lower type t then*

$$\liminf_{r\to R} \frac{w(r, F^{(\alpha,\beta)})}{r(R/R-r)^{\rho+1}} \le \rho t < \rho T \le \limsup_{r\to R} \frac{w(r, F^{(\alpha,\beta)})}{r(R/R-r)^{\rho+1}}. \tag{3.5}$$

PROOF. The proof proceeds exactly on the lines of Lemma 1, hence details are omitted.

DEFINITION. *A real valued GBASP $F^{(\alpha,\beta)}$ is said to be of regular growth if $0<\lambda=\rho<\infty$ and further, it is of perfactly regular growth if $t=T$.*

A real valued GBASP $F^{(\alpha,\beta)}$ which are not of regular growth are called of irregular growth.

LEMMA 3. *The lower type of a real valued GBASP $F^{(\alpha,\beta)}$ of irregular growth is zero.*

PROOF. If $F^{(\alpha,\beta)}$ is of irregular growth than $\rho>\lambda>0$. We have

$$\liminf_{r\to R} \frac{\log^+\log^+ M(r, F^{(\alpha,\beta)})}{\log(R/R-r)} = \lambda.$$

Since $M(r, F^{(\alpha,\beta)})\to\infty$ as $r\to R$, $\log^+$ may be replaced by log. For given $\varepsilon>0$ and $r>r_0(\varepsilon)$,

$$\log M(r, F^{(\alpha,\beta)}) < (R/R-r)^{\lambda-\varepsilon}. \tag{3.6}$$

whereas for a sequence of values of $r \to \infty$,

$$\log M(r, F^{(\alpha,\beta)}) < (R/R-r)^{\lambda+\varepsilon}. \tag{3.7}$$

Dividing (3.6) and (3.7) by $(R/R-r)^{\rho}$ and passing to limit the argument shows that

$$\liminf_{r \to R} \frac{\log M(r, F^{(\alpha,\beta)})}{\log(R/R-r)} = 0.$$

From Lemma 3, we conclude that $t > 0$ is only limited to the study of a GBASP $F^{(\alpha,\beta)}$ of regular growth. In such case we define

$$\liminf_{r \to R} \frac{\log M(r, F^{(\alpha,\beta)})}{\log(R/R-r)^{\lambda}} = t_{\lambda}.$$

The quanty t_{λ} is termed as $\lambda-$ type of a real valued GBASP $F^{(\alpha,\beta)}$. It is significant to mention that there exist a GBASP $F^{(\alpha,\beta)}$ for which t_{λ} is nonzero and finite. For such $F^{(\alpha,\beta)}$, we shall utilise the comparision function $F^{(\alpha,\beta)}$ analogous to proximate type as $\lambda-$ proximate type $S_{\lambda}(r)$. The significance of $S_{\lambda}(r)$ is justified in Theorem 4.

THEOREM 3. *Let a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hypersphere $\sum_R^{(\alpha,\beta)}$ having order ρ $(0 < \rho < \infty)$ and type T $(0 \le T \le \infty)$ such that limits in* (3.2) *and* (3.5) *exist. Then, for a positive real number η, $\log(\eta^{-1} M(r, F^{(\alpha,\beta)})/(R/R-r)^{\rho}$ is a proximate type of a GBASP $F^{(\alpha,\beta)}$.*

PROOF. For a given constant η $(0 < \eta < \infty)$ let

$$S_{\rho}(r) = \frac{\log(\eta^{-1} M(r, F^{(\alpha,\beta)}))}{\log(R/R-r)^{\rho}}. \tag{3.8}$$

Since $\log M(r, F^{(\alpha,\beta)})$ is positive, continuous and increasing function of r for $r > r_0 > 0$, which is differentiable in adjacent open intervals, it follows that $S_{\rho}(r)$ satisfies (iii). Existance of limit in (3.5) implies that $F^{(\alpha,\beta)}$ is of perfectly regular growth and moreover, $S_{\rho}(r) \to T$ as $r \to R$.

Differentiating (3.8), we get

$$\frac{S'_{\rho}(r)}{S_{\rho}(r)} = \frac{M(r, F^{(\alpha,\beta)})}{M(r, F^{(\alpha,\beta)}) \log(\eta^{-1} M(r, F^{(\alpha,\beta)}))} = \frac{\rho R}{R-r},$$

so that

$$(3.9)\qquad \frac{(R-r)\,S'_\rho(r)}{R\,S_\rho(r)} = \frac{(R-r)\,w(r,F^{(\alpha,\beta)})}{R\,r\log(\eta^{-1}M(r,F^{(\alpha,\beta)}))} - \rho\,.$$

Again, limits in (3.2) exist by assumption, hence

$$\frac{(R-r)\,S'_\rho(r)}{R\,S_\rho(r)} \to 0 \quad \text{as} \quad r \to R.$$

Thus $S_\rho(r)$ satisfies the conditiond (ii).

From (3.8), (iv) is readily obtained. In this way all the assertions for $S_\rho(r)$ to be a proximate type of GBASP $F^{(\alpha,\beta)}$ are satisfied and hence the theorem.

THEOREM 4. *Let a real valued GBASP* $F^{(\alpha,\beta)}$ *regular in open hypersphere* $\sum_R^{(\alpha,\beta)}$ *and having order* ρ, *lower order* λ $(0<\lambda\le\rho<\infty)$, *type T and lower type t. Then*

$$(3.10)\qquad \begin{matrix} c/t \\ d/t \end{matrix} \;\le\; \lim_{r\to R} \frac{(R-r)\,S'_\rho(r)}{R\,S_\rho(r)} + \rho \;\le\; \frac{c}{t}\,,$$

where

$$(3.11)\qquad \lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\; \frac{w(r,F^{(\alpha,\beta)})1}{r(R/R-r)^{\rho+1}} = \frac{c}{d}\,,$$

Moreover, if $F^{(\alpha,\beta)}$ is of irregular growth then

$$(3.12)\qquad -\infty \;\le\; \lim_{r\to R} \frac{(R-r)\,S_\lambda(r)}{R\,S(r)} \;\le\; \frac{d}{t_\lambda} - \lambda\,,$$

where $S_\lambda(r)$ is a function in (3.8) corresponding to λ and t_λ is the $\lambda-$ type of $F^{(\alpha,\beta)}$.

PROOF. By (3.1) and the definition of type T and lower type t we obseve that

$$(3.13)\qquad \lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\; \frac{1}{(R/R-r)^{\rho}} \int_{r_0}^{r} \frac{w(r,F^{(\alpha,\beta)})}{x}\,dx = \frac{T}{t},$$

Similarly, for GBASP $F^{(\alpha,\beta)}$ of irregular growth,

$$(3.14)\qquad \lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\; \frac{1}{(R/R-r)^{\rho}} \int_{r_0}^{r} \frac{w(r,F^{(\alpha,\beta)})}{x}\,dx = t_\lambda\,.$$

Fix $r_0 \in [0,\infty)$ such that $\eta = \log M(r_0, F^{(\alpha,\beta)})$. Hence

$$\lim_{r\to R}(\eta^{-1}M(r,F^{(\alpha,\beta)})) = \int_{r_0}^{r}\frac{w(r,F^{(\alpha,\beta)})}{x}dx.$$

Dividing by $(R/R-r)^{\rho}$ and the differentiating with respect to r, we get for almost all values of $r > r_0$,

$$\frac{S'_\rho(r)}{S_\rho(r)} = \frac{w(r,F^{(\alpha,\beta)})}{r\int_{r_0}^{r}\frac{w(x,F^{(\alpha,\beta)})}{x}dx} - \frac{\rho R}{R-r},$$

and this, on simplification, gives (3.9). Now, proceeding to limits in (3.9) and making use of (3.11) and (3.13), the inequalities in (3.10) follows at one.

In case $\rho > \lambda$, we have

$$S_\lambda(r) = \frac{\log(\eta^{-1}M(r,F^{(\alpha,\beta)})}{(R/R-r)^{\lambda}} = \frac{1}{(R/R-r)^{\lambda}}\int_{r_0}^{r}\frac{w(r,F^{(\alpha,\beta)})}{x}dx.$$

By the parallel arguments and making use of (3.14), (3.12) can be disposed of.

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