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COMPUTATION OF MATRIX SPLITTINGS AND THEIR APPLICATIONS

ABSTRACT: A few methods for constructing the index splitting and proper splitting are presented. Also, corresponding representations of the Drazin inverse and the Moore-Penrose inverse are introduced. In partial cases we get known results from [3], [4], [12] and [13]. We also give some convergence criteria for the iterative method for computing the minimal P-norm solution of a given singular linear system $Ax = b$, introduced in [12].

KEY WORDS: index splitting, proper splitting, Drazin inverse, Moore-Penrose inverse, block representation.

1. INTRODUCTION

Let $\mathsf{C}^{m\times n}$ be the set of $m\times n$ complex matrices, and $\mathsf{C}_r^{m\times n}=\{X\in\mathsf{C}^{m\times n}:$ $:\operatorname{rank}(X)=r\}$. The first r columns of A and the first r rows of A we denote by $A^{r|}$ nad $A_{r|}$, respectively. Similarly, $A^{|r}$ nad $A_{|r}$ denote the last r columns of A and the last r rows of A, respectively. By I_r we denote the identity matrix of the order r, and O denotes an appropriate zero block. We use $\mathcal{N}(A)$ to denote the kernel and $\mathcal{R}(A)$ to denote the image of A, and $\rho(A)$ to denote the spectral radius of A. The index of a square matrix A is denoted by $\operatorname{ind}(A)$.

For any matrix $A\in\mathsf{C}^{m\times n}$ consider the following equations in X:

$$(1)\ \ AXA=A,\qquad (2)\ \ XAX=X,\qquad (3)\ \ (AX)^*=AX,\qquad (4)\ \ (XA)^*=XA$$

where the superscript $*$ denotes conjugate and transpose matrix. Also, in the case $m=n$, consider the following equations:

$$(5)\ \ AX=XA \qquad\qquad (1^k)\ \ A^{k+1}X=A^k$$

for a positive integer $k=\operatorname{ind}(A)=\min\{p:\operatorname{rank}(A^{p+1})=\operatorname{rank}(A^p)\}$. For a sequence $\mathcal{S}$ of the elements from the set $\{1,2,3,4,5\}$, the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies the system of equations (1), (2) it is said to be a reflexive g-inverse of A, whereas the

Moore-Penrose inverse $X = A^{\dagger}$ of A satisfies the set of the equations (1)-(4). A matrix $X = A^D$ is said to be the Drazin inverse of A if the equations (1^k), (2) and (5) are satisfied. The group inverse $A^{\#}$ is the unique $\{1, 2, 5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A) = 1$.

Assume that the matrices R, G *are regular*, E, F are permutation matrices and U, V are unitary matrices. Main block decompositions of a given matrix $M \in \mathbb{C}_r^{m\times n}$ are used in [2], [15]:

$$(T_1) \qquad M = R^{-1}\begin{bmatrix} B & O \\ O & O \end{bmatrix} G^{-1} = R^{-1}\begin{bmatrix} I_r \\ O \end{bmatrix} B[I_r, O]\, G^{-1};$$

$$(T_2) \qquad M = U^*\begin{bmatrix} B & O \\ O & O \end{bmatrix} V^* = U^*\begin{bmatrix} I_r \\ O \end{bmatrix} B[I_r, O]\, V^*;$$

$$(T_3) \qquad M = E^*\begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} F^* = E^*\begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}[I_r, T]\, F^*.$$

In the papers [1], [2], [7], [9], [10], [11] there are represented various classes of generalized inverses in terms of block decompositions. In this paper we present a few alternative representations, based on the index splitting and block factorizations $(T_1) - (T_3)$.

We restate the following definition of the index splitting from [12]:

DEFINITION 1.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = k$. *Then the splitting* $A = U - V$ *is called an index splitting provided that*

$$R(U) = R(A^k), \qquad N(U) = N(A^k).$$

In the case $k = 1$ the index splitting reduces to a proper splitting, which is investigated in [3].

The matrix splittings and their applications was investigated by numerous authors [3], [4], [12], [13], [14]. The matrix splittings are useful in representation of generalized inverses as well as in the computation of various iterative solutions of the linear system $Ax = b$. These iterative solutions are based on the general iterative formula $x_{i+1} = x_i + U^- V x_i + U^- b$, where U^- is a kind of generalized inverse of U [13]. The nonsingular case was investigated in [14]. The *proper splitting* and its applications in computation of the Moore-Penrose inverse and the best approximate solution of the linear system was introduced in [3] and [4]. The *index splitting*, which is used in the representation of the Drazin inverse and the minimal P-norm solution was investigated in [12] and [13].

An effective method for constructing the proper splitting was given in [3]. For the construction of the matrix U it is necessary to find a factorization (T_3) for the matrix $M = A$, and then replace the matrix A_{11} by an arbitrary (easily) invertible matrix U_{11} of the order r. Similar metod for constructing the index splitting was introduced in [12]. This method is based on the block decomposition (T_3) of the matrix $M = A^k$, $k = \text{ind}(A)$, and also replaces A_{11} by arbitrary nonsingular matrix U_{11} of the order r. In this paper we are motivated by the idea that it is possible to replace the matrices B in block decompositions (T_1) and (T_2) by an arbitrary invertible matrix H_u.

In the second section we give two methods for construction of the index splitting and the proper splitting. Using these constructions of the matrix splitting we introduce a few methods for computation of the Drazin inverse and the Moore-Penrose inverse. In the third section we introduce several convergence criteria for the iterative method, introduced in [12], which can be used in computation of the minimal P-norm solution of a given singular linear system $Ax = b$, $b \in R(A^k)$, $k = \text{ind}(A) = 1$.

2. Construction of the index splitting and proper splitting

Theorem 2.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be a square matrix satisfying* $\text{ind}(A) = k$ *and* $r = \text{rank}(A^k)$. *Let* H_u *be an arbitrary invertible* $r \times r$ *matrix, and let the matrices* R, G, E, F, U and V *are determined by the application of the corresponding block decompositions* $(T_1) - (T_3)$ *on the matrix* $M = A^k$. *Then the splitting* $A = U_A - V_A$ *is the index splitting of* A *if and only if* U_A *and* V_A *are defined in one of following expressions* (S_i), *where* (S_i) *corresponds to the block decomposition* (T_i) *of the matrix* A^k, $i \in \{1,2,3\}$. *Also, the Drazin inverse of* A *is equal to* $A^D = (I - U_A^{\#}V_A)^{-1}U_A^{\#}$, *where* $U_A^{\#}$ *is defined in* (S_i), $i \in \{1,2,3\}$.

(S_1)
$$U_A = (R^{-1})^{r|} H_u (G^{-1})_{r|} = R^{-1}\begin{bmatrix} H_u & O \\ O & O \end{bmatrix} G^{-1},$$
$$U_A^{\#} = (R^{-1})^{r|}(H_u (G^{-1})_{r|}(R^{-1})^{r|})^{-2} H_u (G^{-1})_{r|},$$

(S_2)
$$U_A = (U^*)^{r|} H_u (V^*)_{r|} = U^*\begin{bmatrix} H_u & O \\ O & O \end{bmatrix} V^*,$$
$$U_A^{\#} = (U^*)^{r|}(H_u (V^*)_{r|}(U^*)^{r|})^{-2} H_u (V^*)_{r|},$$

$$(S_3) \qquad U_A = E^*\begin{bmatrix} I_r \\ S \end{bmatrix} H_u[I_r, T]F^* = E^*\begin{bmatrix} H_u & H_uT \\ SH_u & SH_uT \end{bmatrix}V^*,$$

$$U_A^{\#} = E^*\begin{bmatrix} I_r \\ S \end{bmatrix}\left(H_u[I_r, T](EF)^*\begin{bmatrix} I_r \\ S \end{bmatrix}\right)^{-2} H_u[I_r, T]F^* =$$

$$= E^*\begin{bmatrix} I_r \\ S \end{bmatrix}((EF)^* + T(EF)^*S)^{-1}H_u^{-1}((EF)^* + T(EF)^*S)^{-1}[I_r, T]F^*.$$

PROOF. (S_1) Assume that $A = U_A - V_A$ is the index splitting of A. Since $\mathcal{R}(U_A) = \mathcal{R}(A^k)$ and $\mathcal{N}(U_A) = \mathcal{N}(A^k)$, there exist nonsingular matrices P and Q such that $U_A = PA^k = A^kQ$ (see [3]). Let the matrices P and Q are partitioned as

$$P = R^{-1}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}R, \qquad Q = G\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}G^{-1}.$$

Using block decomposition (T_1) of the matrix A^k, one can verify the following:

$$U_A = R^{-1}\begin{bmatrix} P_{11}B & \mathbb{O} \\ P_{21}B & \mathbb{O} \end{bmatrix}G^{-1} = R^{-1}\begin{bmatrix} BQ_{11} & BQ_{12} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}G^{-1}.$$

A solution of this matrix equation is given by

$$P_{11} = H_uB^{-1}, \quad Q_{11} = B^{-1}H_u, \quad P_{21} = Q_{12} = \mathbb{O}.$$

Hence,

$$U_A = R^{-1}\begin{bmatrix} H_u & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}G^{-1} = (R^{-1})^{r|}H_u(G^{-1})_{r|}.$$

On the other hand, assume that U_A is defined as in part (S_1) of theorem and $V_A = U_A - A$. Then, for the nonsingular matrices

$$P = R^{-1}\begin{bmatrix} H_uB^{-1} & P_{12} \\ \mathbb{O} & P_{22} \end{bmatrix}R, \qquad Q = G\begin{bmatrix} B^{-1}H_u & \mathbb{O} \\ Q_{12} & Q_{22} \end{bmatrix}G^{-1}$$

we have $U_A = PA^k = A^kQ$. Hence, $\mathcal{R}(U_A) = \mathcal{R}(A^k)$, $\mathcal{N}(U_A) = \mathcal{N}(A^k)$, and $A = U_A - V_A$ is the index splitting of A.

Also, $U_A = PQ$, where $P = (R^{-1})^{r|}$, $Q = H_u(G^{-1})_{r|}$, is a full-rank factorization of U_A. According to Cline's general representation of the group inverse [5], we get

$$U_A^{\#} = P(QP)^{-2}Q = (R^{-1})^{r|}(H_u(G^{-1})_{r|}(R^{-1})^{r|})^{-2}H_u(G^{-1})_{r|}.$$

(S_2) Assume that $A = U_A - V_A$ is the index splitting of A and the nonsingular matrices P and Q are partitioned as

$$P = U^*\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}U, \qquad Q = V\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}V^*.$$

Using $U_A = PA^k = A^kQ$ we obtain

$$U_A = U^*\begin{bmatrix} P_{11}B & \mathrm{O} \\ P_{21}B & \mathrm{O} \end{bmatrix}V^* = U^*\begin{bmatrix} BQ_{11} & BQ_{12} \\ \mathrm{O} & \mathrm{O} \end{bmatrix}V^*.$$

Consequently, we have

$$P_{11}B = BQ_{11}, \quad P_{21} = \mathrm{O}, \quad Q_{12} = \mathrm{O}.$$

For example, we can use

$Q_{11} = B^{-1}P_{11}B, \quad P_{21} = \mathrm{O}, \quad Q_{12} = \mathrm{O}, \quad P_{11}, P_{12}, P_{22}, Q_{21}, Q_{22}$ arbitrary.

In this case is

$$U_A = U^*\begin{bmatrix} I_r \\ \mathrm{O} \end{bmatrix}P_{11}B[I_r \ K]V^* = (U^*)^{r|}H_u(V^*)_{r|},$$

where $H_u = P_{11}B$.

On the other hand, assume that U_A is presented in (S_2) and $V_A = A - U_A$. For nonsingular matrices

$$P = U^*\begin{bmatrix} H_uB^{-1} & P_{12} \\ \mathrm{O} & P_{22} \end{bmatrix}U, \qquad Q = V\begin{bmatrix} B^{-1}H_u & \mathrm{O} \\ Q_{12} & Q_{22} \end{bmatrix}V^*$$

we have $U_A = PA^k = A^kQ$. This $\mathcal{R}(U_A) = \mathcal{R}(A^k)$, $\mathcal{N}(U_A) = \mathcal{N}(A^k)$, and $A = U_A - V_A$ is the index splitting of A.

Moreover, $U_A = PQ$, where $P = (U^*)^{r|}$, $Q = H_u(V^*)_{r|}$, is a full-rank factorization of U_A. Again, using the Cline's general representation of the group inverse from [5], we get

$$U_A^{\#} = P(QP)^{-2}Q = (U^*)^{r|}(H_u(V^*)_{r|}(U^*)^{r|})^{-2}H_u(V^*)_{r|}.$$

(S_3) This part of the proof is known from [12] and [13].

REMARK 2.1. In the case $\mathrm{ind}(A)=1$, from Theorem 2.1, we get analogous representations of the group inverse.

Applying the block decompositions $(T_1)-(T_3)$ on the matrix A we get analogous results for the proper splitting and the Moore-Penrose inverse.

THEOREM 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be an arbitrary $m\times n$ matrix of rank r. Let H_u be an invertible $r\times r$ matrix, and let the matrices R, G, E, F, U and V are determined by the corresponding block decompositions $(T_1)-(T_3)$ of the matrix $M=A$. The splitting $A=U_A-V_A$ is the proper splitting of A if and only if U_A is defined in one of the expressions (S_i), corresponding to the block decompositions (T_i) of the matrix A, $i=1,2,3$.*

Also, the Moore-Penrose inverse $A^\dagger$ of A is equal to $A^\dagger=(I-U_A^\dagger V_A)^{-1}U_A^\dagger$, where U_A is defined in the part (S_i) of Theorem 2.1 and $U_A^\dagger$ is equal to the corresponding between the following expressions (M_i), where (M_i) arises from the block decomposition (T_i), $i\in\{1,2,3\}$ of the matrix A.

(M_1) $$U_A^\dagger = (H_u(G^{-1})_{r|})^*(((R^{-1})^{r|})^* A(H_u(G^{-1})_{r|})^*)^{-1}((R^{-1})^{r|})^*$$

(M_2) $$U_A^\dagger = V^{r|}(U_{r|}AV^{r|})^{-1}U_{r|}$$

(M_3) $$U_A^\dagger = F\begin{bmatrix} I_r \\ T^* \end{bmatrix}\left([I_r, S^*]EAF\begin{bmatrix} I_r \\ T^* \end{bmatrix}\right)^{-1}[I_r, S^*]E.$$

PROOF. The representation $A^\dagger=(I-U_A^\dagger V_A)^{-1}U_A^\dagger$ is known from [4]. The results $(M_1)-(M_3)$ follows from the known representation of the Moore-Penrose inverse [2]: if $U_A=PQ$ is a full-rank factorization of U_A, then $U_A^\dagger=Q^*(P^*AQ^*)^{-1}P^*$.

3. COMPUTING MINIMAL P-NORM SOLUTION

In this section we investigate the iterative method for the minimal P-norm solution of the following singular linear system

(3.1) $$Ax=b, \qquad b\in \mathcal{R}(A^k), \quad k=\mathrm{ind}(A)=1.$$

Our investigations are based on the application of the block decompositions (T_1) and (T_2). A method which is based on the block decomposition (T_3) is introduced in [12].

THEOREM 3.1. *Consider the singular linear system* (3.1). *Assume that the matrices R, G, E, F, U and V are determined by the corresponding block decompositions $(T_1)-(T_3)$ of the matrx A. Let H_u be an appropriate invertible $r\times r$ matrix. Assume that $A=U_A-V_A$ is an index splitting of A, and $b\in \mathrm{C}^n$ is an arbitrary vector. Then the following iterative process from* [12]:

$$x_{i+1} = U_A^{\#}V_A x_i + U_A^{\#}b \tag{3.2}$$

converges to $A^D b$, for every $x_0\in \mathrm{C}^n$, if and only if one of the following conditions (K_i) is satisfied, where (K_i) arises from the block decomposition (T_i), $i\in\{1,2,3\}$:

(K_1), (K_2) *The eigenvalues of the matrix $H_u^{-1}B$ are greater than* 0 *and smaller than* 2.

(K_3) *The eigenvalues of $H_u^{-1}A_{11}$ are greater than* 0 *and smaller than* 2.

PROOF. In the cases (K_1) and (K_2) it is not difficult to verify

$$\rho(U_A^{\#}V_A) = \rho(I-H_u^{-1}B). \tag{3.3}$$

For example, the part (K_2) can be verified as follows. Using the part (S_1) of Theorem 2.1 and the block decomposition (T_2) of the matrix A, we get

$$U_A^{\#}V_A = (R^{-1})^{r|}(H_u(G^{-1})_{r|}(R^{-1})^{r|})^{-2}H_u(G^{-1})_{r|}(R^{-1})^{r|}(H_u-B)(G^{-1})_{r|} =$$
$$= (R^{-1})^{r|}((G^{-1})_{r|}(R^{-1})^{r|})^{-1}(I-H_u^{-1}B)(G^{-1})_{r|}.$$

Now, using

$$(G^{-1})_{r|}(R^{-1})^{r|}((G^{-1})_{r|}(R^{-1})^{r|})^{-1} = I,$$

according to Theorem 2 from [3], we get (3.3). Using the known result from [12], iterative method (3.2) converges if and only if

$$\rho(U_A^{\#}V_A) < 1. \tag{3.4}$$

This part of the proof can be completed using (3.4) and (3.3).

Similarly, in the case (K_3) we get

$$\rho(U_A^{\#} V_A) = \rho(I - H_u^{-1} A_{11}),$$

and the proof can be completed using the known principle.

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