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SOME NONLINEAR PROBLEM TO THE EQUATION

 $\Delta u - cu = f$ FOR A SPHERE

ABSTRACT: The subject of the paper is the construction of a solution of the differential equation $\Delta u(x) - c(x)u(x) = f(x, u(x))$, in the spherical domain $D = \{x = (x_1, x_2, x_3) : |x| < R\}$, satisfying the Dirichlet boundary-value condition $u(x) = h(x)$ for $x \in B(D)$. Let $(x, y) \rightarrow G(x, y)$ denote the Green function to the Laplace equation $\Delta G(x, y) = 0$ in the sphere D and to the homogeneous boundary-value condition $G(x, y) = 0$ for $x \in B(D)$, $y \in D$. Applying the change of the unknown function $x \rightarrow u(x) = \bar{u}(x) + w(x)$, where \bar{u} is a solution to the equation $\Delta \bar{u}(x, y) = 0$, $x \in D$, with the boundary-value condition $\bar{u}(x) = h(x)$, $x \in B(D)$, and w is a new unknown function, we obtain the equation $\Delta w(x) = \Delta(u(x) - \bar{u}(x)) = c(x)(\bar{u}(x) + w(x)) + f(x, \bar{u}(x) + w(x))$ or the equation $\Delta w(x) = F_1(x, w(x) = c(x)(\bar{u}(x) + F(x, w(x))$ with $F(x, w(x)) = c(x)w(x) + f(x, \bar{u}(x) + w(x))$, where $x \in D$. Inverting the last problem by the Green function G , we obtain the integral equation $w(x) = f_1(x) + \iiint_D F(y(w(y)))G(x, y)dy$, $x \in D$, with $f_1(x) = \iiint_D c(y)\bar{u}(y)G(x, y)dy$, $x \in D$, and the homogeneous boundary-value condition $w(x) = 0$ for $x \in B(D)$. Solving by the Banach fixed point method the last equation we obtain w and $u = \bar{u} + w$.

KEY WORDS: elliptic equation, Dirichlet boundary-value condition.

1. INTRODUCTION

The subject of the paper is the construction of the solution to the nonlinear elliptic equation

$$(1) \quad \Delta u(x) = c(x)u(x) + f(x, u(x)), \quad x \in D,$$

where

$$D = \{x = (x_1, x_2, x_3) : |x| < R\},$$

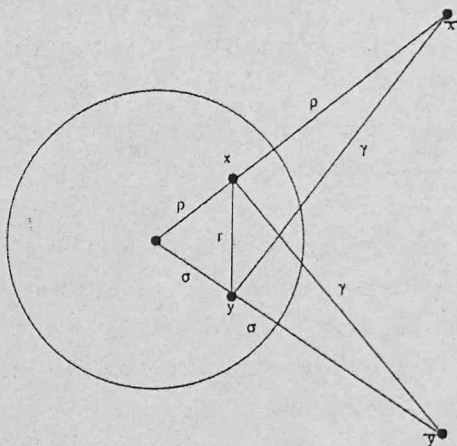
satisfying the boundary-value condition

$$(2) \quad u(x) = h(x), \quad x \in B(D),$$

where f, h are given functions.

Let $(x, y) \rightarrow G(x, y)$ denote the Green function to the differential equation $\Delta \bar{u}(x) = 0$, $x \in D$, and to the boundary-value condition $G(x, y) = 0$ for $x \in B(D)$, $y \in D$. From [1] (p. 252), the function G is of the form:

$$(3) \quad \begin{aligned} G(x, y) &= r^{-1}(x, y) - (R/\rho)^{-1}(\bar{r}(x, y))^{-1} \quad \text{for } x \in D, y \in D, \\ G(x, y) &= r^{-1}(0, y) - R^{-1} \quad \text{for } x = 0, y \in D. \end{aligned}$$



By \bar{x} , \bar{y} we denote the points conjugate to the points x, y with respect to $B(D)$ such that $Ox \cdot O\bar{x} = R^2$.

From formulas (3), we obtain

$$(4) \quad u(x) = \frac{1}{4\pi R} \int \int_{B(D)} \frac{R^2 - \rho^2}{r^3(x, y)} h(y) dS_y, \quad x \in D,$$

with the Poisson kernel

$$K(x, y) = \frac{R^2 - \rho^2}{Rr^3(x, y)}, \quad x \in D, \quad y \in B(D).$$

2. INTEGRAL EQUATION

From differential problem (1), (2) we get the integral equation

$$(5) \quad w(x) = f_1(x) + \iiint_D F(y, w(y)) G(x, y) dy, \quad x \in D,$$

with the boundary-value condition $w(B(D)) = 0$.

3. SOME DEFINITIONS

We shall construct the solution of the integral equation (5) by the Banach fixed point method. At first, we give some definitions. Let (K) denote the class of all functions $x \rightarrow F(x, w(x))$ satisfying the following conditions:

a) the function F is defined and continuous in the set

$$D_1 = \{(x, w) : x \in D, w \in R\},$$

b) the function F is bounded in D_1 i.e. $|F| \leq M$, where $M > 0$ is a constant,

c) the function F satisfies the Lipschitz condition

$$\|F(w_1) - F(w_2)\| < q \|w_1 - w_2\|, \quad w_1, w_2 \in R,$$

uniformly for $x \in \bar{D}$.

Let (K_1) denote the class of all functions $h \in C(B(D_1))$.

Let $c \in C(\bar{D})$, let $c(x) \geq 0$ for $x \in D$ and let B_1 denote the Banach space of continuous functions

$$B_1 = \{x \rightarrow U(x) = c(x)\bar{u}(x)\},$$

with the norm $\|U\| = m = m_1 \cdot m_2$, $m_1 = \sup_{x \in D} c(x)$, $m_2 = \sup |h(x)|$, $x \in B(D)$.

Moreover, let

$$B_2 = \left\{ (x, w) \rightarrow V(x, w) : V(x, w) = \iiint_D F(y, w(y)) G(x, y) dy \right\},$$

with the norm $\|V(w)\| = M_1 = M(1/(4\pi R)) \sup \left| \iiint_D r^{-1}(x, y) dy \right|$, $x \in D$.

Let us, also, consider the space:

$$B^1 = B_1 \times B_2 \text{ with the norm } \|(U, V(w))\| = \|U + V(w)\| \leq \|U\| + \|V\|.$$

By [1] (p. 307), we obtain

$$(6) \quad \|V\| = M2\pi R^2.$$

4. THE SPHERES IN THE BANACH SPACES

Let $\bar{0}$ denote the vector function with coordinates identically equal zero and let $R_1 = m + M2\pi R^2$.

Let us consider three balls:

The ball $K(\bar{O}, R_1)$ with the center \bar{O} and radius R_1 being the set of all functions $(U, V(x, w))$ such that $\|(U, V(w))\| < R_1$; the ball $K(\bar{O}, qR_1)$ with the center \bar{O} and radius qR_1 being the set of the functions U such that $\|U\| = m < qR_1$; the ball with the center \bar{O} and radius $(1-q)R_1$ being the set of all functions $V(w)$ such that $2M\pi R^2 < (1-q)R_1$.

5. TRANSFORMATION TO THE BANACH FIXED POINT METHOD

Let us consider the transformation

$$(S) \quad x \rightarrow S(x, U(x), V(x, w)) = U(x) + V(x, w), \quad x \in \bar{D}.$$

THEOREM 1. *If $q \in (0, 1)$ then the transformation (S) satisfies the conditions:*

1^o *S is the contraction,*

2^o *S transforms the ball $K(\bar{O}, R_1)$ into itself.*

PROOF. 1^o: We have

$$\begin{aligned} \|S(U, w_1) - S(U, w_2)\| &= \|V(w_1) - V(w_2)\| \leq \\ &\leq q \|w_1 - w_2\|, \quad w_1, w_2 \in R. \end{aligned}$$

$$2^o \quad \|(U, V(w))\| = \|U + V(w)\| \leq \|U\| + \|V(w)\| \leq qR_1 + (1-q)R_1 = R_1.$$

By the Banach fixed point theorem, we obtain:

THEOREM 1. *There exists the fixed point W to the transformation (S) such that*

$$W(x) = U(x) + \iiint_D F(x, W(y)) G(x, y) dy, \quad x \in \bar{D},$$

or

$$(S_1) \quad W(x) = U(x) + V(x, W), \quad x \in \bar{D}.$$

6. CONSTRUCTION OF THE FIXED POINT W

To construct the fixed point W , we shall apply the method of the successive approximations. Let us consider the sequence

$$W_0(x) = U(x), \quad x \in J,$$

where U is an arbitrary point of the ball $K(\bar{O}, QR_1)$,

$$W_1(x) = W_0(x) + V(x, W_0), \quad x \in \bar{D},$$

$$\dots\dots\dots$$

$$W_n(x) = W_0(x) + V(x, W_{n-1}), \quad n = 1, 2, \dots$$

LEMMA 1. *If $F \in (K)$ and $q \in (0, 1)$ then*

$$\|W_n - W_m\| < q^m (1 - q)^{-1} \|W_1 - W_0\|, \quad m, n = 1, 2, \dots, \quad n > m.$$

PROOF. We have

$$\|W_n - W_{n-1}\| < q \|W_{n-1} - W_{n-2}\| \leq \dots \leq q^n \|W_1 - W_0\|.$$

Let us consider the Cauchy sequence

$$W_{n,m}(x) = W_n(x) - W_m(x), \quad n > m, \quad m, n = 1, 2, \dots, \quad x \in \bar{D}.$$

We can write

$$W_n(x) - W_m(x) = \sum_{j=m+1}^n (W_j(x) - W_{j-1}(x))$$

By the last formula, we obtain

$$\begin{aligned} \|W_{n,m}(x)\| &\leq \sum_{j=m+1}^n q^{j+1} \|W_1 - W_0\| \leq \sum_{j=0}^{\infty} q^{j+1} \|W_1 - W_0\| \leq \\ &\leq q^m (1 - q)^{-1} \|W_1 - W_0\| \end{aligned}$$

for arbitrary positive integers $n, m > N$. The sequence $W_{n,m}$ of continuous functions W_n is a complete space. Hence, there exists

$$\lim_{n \rightarrow \infty} W_n(x) = W(x), \quad x \in \bar{D}, \quad W \in C(\bar{D}),$$

and, finally, we obtain the function W satisfying equation (S_1)

$$W(x) = f_1(x) + V(x, W), \quad x \in \bar{D},$$

which is the solution of the differential problem (1), (2). By the Banach fixed point theorem, the solution of equation (S_1) is unique.

7. THE UNIQUENESS THEOREM

Applying the fundamental formula for the differential problem

$$\Delta W(x) = c(x)\bar{u}(x) + F(x, W(x)), \quad x \in D, \quad w(B(D)) = 0,$$

we obtain

$$(8) \quad W(x) = \iiint_D c(y)\bar{u}(y)G(x, y)dy + \\ + \iiint_D F(y, w(y))G(x, y)dy, \quad x \in D, \quad w(B(D)) = 0.$$

Thus, from the differential problem we have integral equation (5). Conversely, since the function $y \rightarrow c(y)\bar{u}(y) + F(y, w(y)) = F^1(y)$ is Lipschitz continuous then, applying the Poisson formula [1] (p.326), we obtain $\Delta w(x) = F^1(x)$, $x \in D$, or

$$\Delta w(x) = C(x)\bar{u}(x) + F(x, w(x)), \quad x \in D, \quad w(B(D)) = 0.$$

Hence, the differential problem (1), (2) is equivalent to equation (5) for which the solution w is unique. Also, the solution \bar{u} is unique and, consequently, we obtain:

THEOREM 2. *The function $u = \bar{u} + w$ is the unique solution to problem (1), (2).*

REFERENCES

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