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ON AN HYPERBOLIC FUNCTIONAL DIFFERENTIAL INCLUSION IN BANACH SPACES

ABSTRACT: In this paper we investigate the existence of solutions to an hyperbolic functional differential inclusion in Banach spaces. We shall rely on a fixed point theorem for condensing maps due to of Martelli.

KEY WORDS: hyperbolic functional differential inclusion, convex valued multivalued map, existence, condensing map, fixed point.

1. INTRODUCTION

This note deals with the existence of solutions for the following hyperbolic functional differential inclusion (Darboux problem):

$$(1.1)\qquad \frac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u_{(x,y)}), \qquad (x,y)\in J_a \times J_b = [0,a]\times[0,b]$$

$$(1.2)\qquad u(x,y)=\phi(x,y), \qquad (x,y)\in[-r_1,a]\times[-r_2,b]\setminus(0,a]\times(0,b]$$

where $F: J_a \times J_b \times C([-r_1,0]\times[-r_2,0],E)\to 2^E$ is a closed, bounded and convex valued multivalued map, $\phi\in C([-r_1,a]\times[-r_2,b]\setminus(0,a]\times(0,b],E)$, $a>0,\ b>0,\ r_1>0,\ r_2>0$ and $(E,|\cdot|)$ a separable Banach space.

For each $u\in C([-r_1,a]\times[-r_2,b],E)$ and each $(x,y)\in J_a\times J_b$ the function $u_{(x,y)}:[-r_1,0]\times[-r_2,0]\to E$ is defined by

$$u_{(x,y)}(s,t) = u(x+s,y+t), \quad \text{for each} \quad (s,t)\in[-r_1,0]\times[-r_2,0].$$

In the last three decades several papers have been devoted to the study of partial functional differential equations. The methods used are the iterative methods [2], [5]; Chaplyghin method [6], [13]; difference method [4], [14]. Other existence results for the Darboux problem for functional differential equations can be found in the recent monograph [12] and in the papers [3], [10], [19] and for non-functional partial differential equations in [7], [8], [15], [18].

In this note we shall give an existence result for the problem (1.1) – (1.2). The method we are going to use to reduce the existence of solutions to problem (1.1) – (1.2) to the search for fixed points of a suitable multivalued map on the Banach space $C([-r_1,a]\times[-r_2,b],E)$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for condensing maps due to Martelli [17]. Our result, extends the above results to the multivalued case, and moreover, complements the few existence results on the problem.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this paper. In the sequel we will denote by $C([-r_1,0]\times[-r_2,0],E)$ the Banach space of continuous functions from $[-r_1,0]\times[-r_2,0]$ into E with the usual supremum norm $\|\cdot\|$ and by $C(J_a\times J_b,E)$ the Banach space of continuous functions from $J_a\times J_b$ into E with the norm

$$\|z\|_\infty := \sup\{|z(x,y):(x,y)\in J_a\times J_b\}, \qquad \text{for each} \quad z\in C(J_a\times J_b,E).$$

A measurable function $z: J_a\times J_b\to E$ is Bochner integrable if and only if $|z|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [20]). $L^1(J_a\times J_b,E)$ denotes the Banach space of measurable functions $z: J_a\times J_b\to E$ which are Bochner integrable.

Let $(X,|\cdot|)$ be a Banach space. A multivalued map $G:X\to 2^E$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. G is bounded on bounded sets if $G(B)=\bigcup_{x\in B}G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y|:y\in G(x)\}\}<\infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_*\in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V)\subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B\subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n\to x_*$, $y_n\to y_*$, $y_n\in G(x_n)$ imply $y_*\in G(x_*)$). G has a fixed point if there is $x\in X$ such that $x\in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G:J_a\times J_b\times C([-r_1,0]\times[-r_2,0],E)\to BCC(E)$ is said to be measurable if for each $w\in E$ the function $Y:J_a\times J_b\to \mathrm{IR}$ defined by

$$Y(x,y)=d(w,G(x,y,u))=\inf\{|w-v|:v\in G(x,y,u)\}$$

is measurable.

An upper semi-continuous nap $G: X \to 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel [1].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [9] and Hu and Papageorgiou [11].

DEFINITION 2.1. *A multivalued map* $J_a \times J_b \times C([-r_1,0]\times[-r_2,0],E) \to 2^E$ *is said to be an* L^1 *– Carathéodory if*

(i) $(x,y) \mapsto F(x,y,u)$ is measurable for each $u \in C([-r_1,0]\times[-r_2,0],E)$;

(ii) $u \mapsto F(x,y,u)$ *is upper semicontinuous for almost all* $(x,y) \in J_a \times J_b$;

(iii) *For each* $k > 0$, *there exists* $h_k \in L^1(J_a \times J_b, \mathrm{IR}_+)$ *such that*

$$\| F(x,y,u) \| = \sup\{|v| : v \in F(x,y,u)\} \leq h_k(t) \text{ for all } \|u\| \leq k$$

and for almost all $(x,y) \in J_a \times J_b$.

We will need the following hypotheses:

(H1) $F: J_a \times J_b \times C([-r_1,0]\times[-r_2,0],E) \to BCC(E)$ is an L^1 – Carathéodory multivalued map and for each fixed $u \in C([-r_1,a]\times[-r_2,b],E)$ the set

$$S_{F,u} = \{v \in L^1(J_a \times J_b, E) : v(x,y) \in F(x,y,u_{(x,y)}) \text{ for a.e. } (x,y) \in J_a \times J_b\}$$

is nonempty;

(H2) There exist functions $p,q \in L^1(J_a \times J_b, \mathrm{IR}_+)$ such that

$$\| F(x,y,u) \| := \sup\{|v| : v \in F(x,y,u)\} \leq p(x,y) + q(x,y)\| u_{(x,y)} \|$$

for almost all $(x,y) \in J_a \times J_b$ and all $u \in C([-r_1,a]\times[-r_2,b],E)$;

(H3) For each bounded set $B \subset C([-r_1,a]\times[-r_2,b],E)$ and for each $(x,y) \in J_a \times J_b$ the set

$$\left\{\phi(x,0) + \phi(0,y) - \phi(0,0) + \int_0^x \int_0^y v(t,s)\,dt\,ds \; : \; v \in S_{F,B}\right\}$$

is relatively compact in E, where $S_{F,B} = \bigcup\{S_{F,u} : u \in B\}$.

REMARK 2.2. (i) If $\dim E < \infty$, then for each $u \in C(J_a \times J_b, E)$ the set $S_{F,u}$ is nonempty (see Lasota and Opial [16]).

(ii) If $\dim E = \infty$, then $S_{F,u}$ is nonempty if and only if the function $Y : J_a \times J_b \to \mathrm{IR}^+$ defined by

$$Y(x,y) := \inf\{|v| : v(x,y) \in F(x,y,u_{(x,y)}\}$$

is measurable (see Hu and Papageorgiou [11]).

(iii) (H3) *is satisfied if for each* $(x,y) \in J_a \times J_b$ *the multivalued map* $F(x,y,\cdot)$ *sends bounded sets of* $C([-r_1,0]\times[-r_2,0],E)$ *into relatively compact sets.*

DEFINITION 2.3. *By a solution of* $(1.1)-(1.2)$ *we mean a function* $u(\cdot,\cdot) \in C([-r_1,a]\times[-r_2,b],E)$ *such that, there exists* $v \in L^1(J_a \times J_b, E)$ *for which we have*

$$u(x,y) = \phi(x,0) + \phi(0,y) - \phi(0,0) + \int_0^x\int_0^y v(t,s)dtds \quad \textit{for each } (x,y) \in J_a \times J_b$$

and $v(t,s) \in F(t,s,u(t,s))$ *a.e. on* $J_a \times J_b$ *and* $y(x,y) = \phi(x,y)$ *on* $[-r_1,a]\times[-r_2,b]\setminus(0,a]\times(0,b]$.

Our considerations are based on the following lemmas.

LEMMA 2.4. [16]. *Let F be a multivalued map satisfying* (H1) *and let* Γ *be a linear continuous mapping from* $L^1(J_a \times J_b, E)$ *to* $C(J_a \times J_b, E)$ *the set operator*

$$\Gamma \circ S_F : C(J_a \times J_b, E) \to CC(C(J_a \times J_b, E)), \quad u \mapsto (\Gamma \circ S_F)(u) := \Gamma(S_{F,u})$$

is a closed graph operator in $C(J_a \times J_b, E) \times C(J_a \times J_b, E)$.

LEMMA 2.5. [17]. *Let X be a Banach space and* $N : X \to BCC(X)$ *be a condensing map. If the set*

$$\Omega := \{u \in X : \lambda u \in N(u) \ \textit{for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. MAIN RESULT

Let $D = [-r_1,a]\times[-r_2,b]$ and $\overline{D} = [-r_1,a]\times[-r_2,b]\setminus(0,a]\times(0,b]$. Now, we are able to state and prove our main theorem.

THEOREM 3.1. *Assume that hypotheses* (H1) – (H3) *hold. Then the problem* (1.1) – (1.2) *has at least on solution on D.*

PROOF. Transform the problem (1.1) – (1.2) into a fixed point problem. Consider the multivalued map, $N: C(D,E) \to 2^{C(D,E)}$ defined by:

$$N(u) = \left\{ h \in C(D,E) : h(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \overline{D} \\ \phi(x,0) + \phi(0,y) - \phi(0,0) & \\ \quad + \int_0^x \int_0^y v(t,s)dtds, & (x,y) \in J_a \times J_b \end{cases} \right\}$$

where

$$v \in S_{F,u} = \{ v \in L^1(J_a \times J_b, E) : v(t,s) \in F(t,s,u_{(t,s)}) \text{ for a.e. } (t,s) \in J_a \times J_b \}.$$

REMARK 3.2. It is clear that the fixed points of N are solutions to (1.1) – (1.2).

We shall show that N satisfies the assumptions of Lemma 2.5. The proof will be given in several steps.

Step 1: *$N(u)$ is convex for each $u \in C(J_a \times J_b, E)$.*

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(x,y) \in J_a \times J_b$ we have

$$h_i(x,y) = \phi(x,0) + \phi(0,y) - \phi(0,0) + \int_0^x \int_0^y v_i(t,s)dtds, \qquad i = 1,2.$$

Let $0 \le \alpha \le 1$. Then for each $(x,y) \in J_a \times J_a$ we have

$$(\alpha h_1 + (1-\alpha)h_2)(x,y) = \phi(x,0) + \phi(0,y) - \phi(0,0) + \\ + \int_0^x \int_0^y [\alpha v_1(t,s) + (1-\alpha)v_2(t,s)]ds.$$

Since $S_{F,u}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha)h_2 \in N(u).$$

Step 2: *N is bounded on bounded stes of $C(J_a \times J_b, E)$.*

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(u)$, $u \in B_r = \{u \in C(J_a \times J_b, E) : \| u \|_\infty \leq r\}$ one has $\| h \|_\infty \leq c$.

If $h \in N(u)$, then there exists $v \in S_{F,u}$ such that for each $(x, y) \in J_a \times J_b$ we have

$$h(x, y) = \phi(x,0) + \phi(0, y) - \phi(0,0) + \int_0^x \int_0^y v(t, s)dtds.$$

By (H1) we have, for each $(x, y) \in J_a \times J_b$, that

$$| h(x, y) | \leq | \phi(x,0) | + | \phi(0, y) | + | \phi(0,0) | + \int_0^x \int_0^y h_r(t, s)dtds.$$

Then

$$\| h \|_\infty \leq 3 \| \phi \|_\infty + \int_0^a \int_0^b h_r(t, s)dtds = c.$$

Step 3: *N sends bounded sets of* $C(J_a \times J_b, E)$ *into equicontinuous sets.*

Let $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$, $x_1 < x_2$, $y_1 < y_2$ and B_r be a bounded set of $C(J_a \times J_b, E)$. For each $u \in B_r$ and $h \in N(u)$, there exists $v \in S_{F,u}$ such that

$$h(x, y) = \phi(x,0) + \phi(0, y) - \phi(0,0) + \int_0^x \int_0^y v(t, s)dtds.$$

Thus we obtain

$$\begin{aligned} | h(x_2, y_2) - h(x_1, y_1) | &\leq | \phi(x_2,0) - \phi(x_1,0) | + | \phi(0, y_2) - \phi(0, y_1) | + \\ &+ \int_{x_1}^{x_2} \int_{y_1}^{y_2} | v(t, s) | dtds \leq \\ &\leq | \phi(x_2,0) - \phi(x_1,0) | + | \phi(0, y_2) - \phi(0, y_1) | + \\ &+ \int_{x_1}^{x_2} \int_{y_1}^{y_2} h_r(t, s) | dtds. \end{aligned}$$

As $(x_2, y_2) \to (x_1, y_1)$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H3), together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous, and therefore, a condensing multivalued map.

Step 4: *N has a closed graph.*

Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that

$$h_n(x,y) = \phi(x,0) + \phi(0,y) - \phi(0,0) + \int_0^x \int_0^y v_n(t,s)dtds, \quad (x,y) \in J_a \times J_b.$$

We must prove that there exists $g_* \in S_{F,u_*}$ such that

$$h_*(x,y) = \phi(x,0) + \phi(0,y) - \phi(0,0) + \int_0^x \int_0^y v_*(t,s)dtds, \quad (x,y) \in J_a \times J_b.$$

Now, we consider the linear continuous operator

$$\Gamma : L^1(J_a \times J_b, E) \to C(J_a \times J_b, E)$$

$$v \mapsto \Gamma(v)(x,y) = \int_0^x \int_0^y v(t,s)dtds, \quad (x,y) \in J_a \times J_b.$$

From Lemm 2.4, it follows that $\Gamma \circ S_F$ is a closed graph operator. Clearly we have

$$\|(h_n(x,y) - \phi(x,0) - \phi(0,y) + \phi(0,0)) - (h_*(x,y) - \phi(x,0) - \phi(0,y) + \phi(0,0))\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover from the definition of Γ we have

$$(h_n(x,y) - \phi(x,0) - \phi(0,y) + \phi(0,0)) \in \Gamma(S_{F,u_n}).$$

Since $u_n \to u_*$, it follows from Lemma 2.4 that

$$h_*(x,y) - \phi(x,0) + \phi(0,y) - \phi(0,0) = \int_0^x \int_0^y v_*(t,s)dtds, \quad (x,y) \in J_a \times J_b$$

for some $v_* \in S_{F,u_*}$.

Step 5: *The set*

$$\Omega := \{u \in C(J_a \times J_b, E) : \lambda u \in N(u) \ \textit{for some} \ \lambda > 1\}$$

is bounded.

Let $u \in \Omega$. Then $\lambda u \in N(u)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,u}$ such that

$$u(x,y) = \lambda^{-1}\phi(x,0) + \lambda^{-1}\phi(0,y) - \lambda^{-1}\phi(0,0) + \lambda^{-1}\int_0^x\int_0^y v(t,s)dtds,$$

$$(x,y) \in J_a \times J_b.$$

This implies, by (H2), that for each $(x,y) \in J_a \times J_b$ we have

$$|u(x,y)| \le |\phi(x,0)| + |\phi(0,y)| + |\phi(0,0)| + \int_0^x\int_0^y (p(t,s) + q(t,s)\|u_{(t,s)}\|)dtds.$$

We consider the function μ defined by

$$\mu(x,y) = \sup\{|y(t,s)| : (t,s) \in [-r_1,x]\times[-r_2,y]\}, \quad (x,y) \in J_a \times J_b.$$

Let $(x^*,y^*) \in [-r_1,x]\times[-r_2,y]$ be such that $\mu(x,y) = |y(x^*,y^*)|$. If $(x^*,y^*) \in J_a \times J_b$, by the previous inequality, we have for $(x,y) \in J_a \times J_b$

$$\mu(x,y) \le |\phi(x,0)| + |\phi(0,y)| + |\phi(0,0)| + \int_0^x\int_0^y (p(t,s) + q(t,s)\|u_{(t,s)}\|)dtds \le$$

$$\le 3\|\phi\| + \int_0^x\int_0^y (p(t,s) + q(t,s)\mu(t,s))dtds \le$$

$$\le 3\|\phi\| + \|p\|_{L^1(J_a\times J_b)} + \int_0^x\int_0^y q(t,s)\mu(t,s)dtds.$$

Invoking Gronwall's inequality we get that

$$\mu(x,y) \le [3\|\phi\| + \|p\|_{L^1(J_a\times J_b)}]\exp\|q\|_{L^1(J_a\times J_b)} = M.$$

Since for every $(x,y) \in J_a \times J_b$, $\|u_{(x,y)}\| \le \mu(x,y)$, we have

$$\|u\|_\infty := \sup\{|u(x,y)| : (x,y) \in D\} \le M.$$

This shows that Ω is bounded.

Set $X := C(D,E)$. As a consequence of Lemma 2.5 we deduce that N has a fixed point which is a solution of (1.1) – (1.2) on D.

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