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BOUNDED OSCILLATION FOR A CLASS OF EVEN ORDER NEUTRAL DIFFERENCE EQUATIONS*

ABSTRACT: We investigate bounded oscillation for the even order neutral delay difference equation

$$\Delta^u (x_n - cx_{n-m}) = p_n x_{n-k},$$

where u is even. The sufficient conditions obtained in this paper improve and generalize the results in related literature.

KEY WORDS: neutral difference equation, bounded solution, oscillation, nonoscillation.

1. INTRODUCTION

Recently, there has been a lot of activity concerning the oscillatory behavior of difference equations, and various applications have been found in the literature. We refer to [1-9] and the references cited therein for more details.

In [3, 4], the authors considered the following second order neutral delay difference equation

$$\Delta^2 (x_n - cx_{n-m}) = p_n x_{n-k}, \qquad n \ge n_0, \tag{1}$$

and proved that Eq. (1) always has an unbounded positive solution, where c, p_n are real numbers with $p_n \ge 0$, $p_n \not\equiv 0$, $n \ge n_0$, m, k, n_0 are nonnegative integers and $m \ge 1$, Δ denotes the forward operator $\Delta x_n = x_{n+1} - x_n$. Therefore, for Eq. (1) we only need to find conditions for all bounded solutions to be oscillatory. [3] first established such conditions in the cases when $0 < c < 1$ and $c > 1$. Later, these conditions were further improved by [4]. The main results in [4] are the following two theorems.

THEOREM A. *Let* $0 \le c < 1$ *and* $k \ge 1$. *If*

$$\limsup_{n\to\infty} \sum_{i=0}^{k-1} (i+1) p_{n+i} > 1 - c, \tag{2}$$

then every bounded solution of Eq. (1) *oscillates.*

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THEOREM B. *Let* $c<0$ *and* $k>m$. *If*

(3) $$\limsup_{n\to\infty} \frac{p_n}{p_{n-m}} = \alpha \in (0,\infty)$$

and

(4) $$\limsup_{n\to\infty} \sum_{i=1}^{k-m} i\cdot p_{n+i} > 1-c\alpha,$$

then every bounded solution of Eq. (1) *oscillates.*

In this paper, we consider the following more general even order neutral delay difference equation

(5) $$\Delta^u (x_n - cx_{n-m}) = p_n x_{n-k},$$

where $u\geq 2$ is an even integer, c, p_n, m and k are the same as in (1). Our main aim is to establish some criteria which guarantee every bounded solution of Eq. (5) oscillates, which generalize and improve the above Theorem A and Theorem B.

For the sake of convenince, throughout this paper, we use the convention

$$\sum_{n=i}^{j} p_n \equiv 0 \quad \text{whenever} \quad j\leq i-1,$$

$$x^{(0)}=1 \quad \text{and} \quad x^{(n)} = x(x-1)(x-2)\cdots(x-(n-1)) = \prod_{j=0}^{n-1}(x-j).$$

2. MAIN RESULTS

THEOREM 1. *Assume that* $0\leq c<1$ *and* $k\geq 1$, *and that*

(6) $$\limsup_{n\to\infty} \sum_{i=0}^{k-1} \sum_{j=n}^{n+k-i} \frac{q_i(j)}{1-c+q_{k+1}(j)} \left(\prod_{s=j-k+i+1}^{j+1} \frac{1-c+q_{k+1}(s-1)}{1-c-q_k(s-1)} \right) > 1,$$

where $q_i(n) = ((i+u-2)^{(u-2)}/(u-2)!)p_{i+n}$. *Then every bounded solution of Eq.* (5) *oscillates.*

PROOF. Suppose the contrary, and let $\{x_n\}$ be and bounded eventually positive solution of (5). Set $y_n = x_n - cx_{n-m}$. Then $\{y_n\}$ is bounded and $\Delta^u y_n = p_n x_{n-k} \geq 0$. It follows that $\Delta^i y_n$ $(i=1,2,...,u-1)$ are monotone and each of them doesn't change sign eventually. In view of [1. Theorem 1.7.11], there exists an integer $n_1 > n_0$ such that

(7) $(-1)^i \Delta^i y_n > 0$ for $n \geq n_1$ and $\lim\limits_{n\to\infty} \Delta^i y_n = 0, \quad i = 0, 1, 2, \cdots, u-1.$

Let $h_n = [(n-n_1)/m]$, where $[\cdot]$ denotes the greatest integer function. Then we have for $n > n_1$

$$x_n = y_n + cx_{n-m} =$$
$$= y_n + cy_{n-m} + c^2 y_{n-2m} + \cdots + c^{h_n-1} y_{n-(h_n-1)m} + c^{h_n} x_{n-h_n m}.$$

By using the decreasing nature of $\{y_n\}$, we have

$$x_n \geq (1 + c + c^2 + \cdots + c^{h_n-1}) y_n = \frac{1-c^{h_n}}{1-c} y_n.$$

From (6) there exists sufficiently small positive number ε such that

$$\limsup_{n\to\infty} \sum_{i=0}^{k-1} \sum_{j=n}^{n+k-i} \frac{(1-\varepsilon)q_i(j)}{1-c+(1-\varepsilon)q_{k+1}(j)} \times \\ \times \left(\prod_{s=j-k+i+1}^{j+1} \frac{1-c+(1-\varepsilon)q_{k+1}(s-1)}{1-c-(1-\varepsilon)q_k(s-1)} \right) > 1. \tag{8}$$

For this ε, there must exists an integer N such that

$$\frac{1-c^{h_n}}{1-c} \geq \frac{1-\varepsilon}{1-c}, \qquad n \geq N-k,$$

and thus

$$x_n \geq \frac{1-\varepsilon}{1-c} y_n, \qquad n \geq N-k.$$

Substituting the last inequality into Eq. (5), we have

$$\Delta^n y_n \geq \frac{1-\varepsilon}{1-c} p_n y_{n-k}, \qquad n \geq N. \tag{9}$$

Summing (9) from $n \geq N$ to ∞ for $(u-1)$ times and using (7), we have

$$\Delta y_n + \frac{1-\varepsilon}{1-c} \sum_{i=n}^{\infty} \frac{(i-n+u-2)^{(u-2)}}{(u-2)!} p_i y_{i-k} \leq 0.$$

Hence

$$\Delta y_n + \frac{1-\varepsilon}{1-c} \sum_{i=n}^{n+k+1} \frac{(i-n+u-2)^{(u-2)}}{(u-2)!} p_i y_{i-k} \leq 0,$$

and thus

$$\Delta y_n + \frac{1-\varepsilon}{1-c}\sum_{i=0}^{k+1}\frac{(i+u-2)^{(u-2)}}{(u-2)!}p_{i+n}y_{i+n-k} \le 0.$$

That is

(10)
$$\Delta y_n + \frac{1-\varepsilon}{1-c}\sum_{i=0}^{k+1} q_i(n)y_{i+n-k} \le 0, \qquad n \ge N.$$

From (10) we have

$$y_{n+1} - y_n + \frac{1-\varepsilon}{1-c}q_i(n)y_{i+n-k} \le 0, \quad 0 \le i \le k+1.$$

Then

$$y_{n+1} - y_n + \frac{1-\varepsilon}{1-c}q_i(n)y_n \le 0, \qquad 0 \le i \le k.$$

Hence

$$0 < y_{n+1} \le \left(1 - \frac{1-\varepsilon}{1-c}q_i(n)\right)y_n,$$

which implies

(11)
$$1-c-(1-\varepsilon)q_i(n) > 0, \qquad 0 \le i \le k.$$

On the other hand, it follows from (10) that

$$\left(1+\frac{1-\varepsilon}{1-c}q_{k+1}(n)\right)y_{n+1} - \left(1-\frac{1-\varepsilon}{1-c}q_k(n)\right)y_n + \\ + \frac{1-\varepsilon}{1-c}\sum_{i=0}^{k-1} q_i(n)y_{i+n-k} \le 0, \quad n \ge N.$$

Therefore, we have

$$y_{n+1} - \frac{1-c-(1-\varepsilon)q_k(n)}{1-c+(1-\varepsilon)q_{k+1}(n)}y_n + \\ + \frac{1-\varepsilon}{1-c+(1-\varepsilon)q_{k+1}(n)}\sum_{i=0}^{k-1} q_i(n)y_{i+n-k} \le 0, \quad n \ge N.$$

Multpling both sides of the last inequality by $\prod_{i=N}^{n+1}\frac{1-c+(1-\varepsilon)q_{k+1}(i-1)}{1-c-(1-\varepsilon)q_k(i-1)}$, we have

$$\Delta\left(y_n\prod_{i=N}^{n}\frac{1-c+(1-\varepsilon)q_{k+1}(i-1)}{1-c-(1-\varepsilon)q_k(i-1)}\right) + \\ + \sum_{i=0}^{k-1}\left(y_n\prod_{j=N}^{n+1}\frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)}\right)\frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)}y_{n-k+i} \le 0.$$

Set $z_n = y_n \prod_{i=N}^{n} \frac{1-c+(1-\varepsilon)q_{k+1}(i-1)}{1-c-(1-\varepsilon)q_k(i-1)}$, then we obtain

$$\Delta z_n + \sum_{i=0}^{k-1} \left(\prod_{j=N}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \times$$

$$\times \frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)} z_{n-k+i} \prod_{j=N}^{n-k+1} \frac{1-c-(1-\varepsilon)q_k(j-1)}{1-c+(1-\varepsilon)q_{k+1}(j-1)} \le 0.$$

So, we have

$$\Delta z_n + \sum_{i=0}^{k-1} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \times \tag{12}$$

$$\times \frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)} z_{n-k+i} \le 0.$$

It is obvious that (12) has an eventually positive solution and $\Delta z_n < 0$. Summing (12) from M to ∞, we have

$$z_M \ge \sum_{n=M}^{\infty} \frac{1}{1-c+(1-\varepsilon)q_{k+1}(n)} \sum_{i=0}^{k-1} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \times$$

$$\times (1-\varepsilon)q_i(n) z_{n-k+i} \ge$$

$$\ge \sum_{i=0}^{k-1} \sum_{n=M}^{M+k-i} \frac{1}{1-c+(1-\varepsilon)q_{k+1}(n)} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \times$$

$$\times (1-\varepsilon)q_i(n) z_{n-k+i} \ge$$

$$\ge z_M \sum_{i=0}^{k-1} \sum_{n=M}^{M+k-i} \frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right).$$

Thus, we have

$$\sum_{i=0}^{k-1} \sum_{n=M}^{M+k-i} \frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \le 1. \tag{13}$$

Taking the limit superior as $M \to \infty$ we obtain

$$\limsup_{M\to\infty} \sum_{i=0}^{k-1} \sum_{n=M}^{M+k-i} \frac{(1-\varepsilon)q_i(n)}{1-c+(1-\varepsilon)q_{k+1}(n)} \left(\prod_{j=n-k+i+1}^{n+1} \frac{1-c+(1-\varepsilon)q_{k+1}(j-1)}{1-c-(1-\varepsilon)q_k(j-1)} \right) \le 1.$$

That is

$$\limsup_{M\to\infty}\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}\frac{(1-\varepsilon)q_i(j)}{1-c+(1-\varepsilon)p_{k+1}(j)}\times \tag{14}$$
$$\times\left(\prod_{s=j-k+i+1}^{j+1}\frac{1-c+(1-\varepsilon)q_{k+1}(s-1)}{1-c-(1-\varepsilon)q_k(s-1)}\right)\le 1.$$

This contradicts (8). The proof is complete.

When $u=2$, it is easy to see that $q_i(n)=p_{n+i}$. So we obtain the following.

COROLLARY 1. *Assume that* $0\le c<1$ *and* $k\ge 1$ *and that*

$$\limsup_{n\to\infty}\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}\frac{p_{i+j}}{1-c+p_{k+1+j}}\left(\prod_{j=n-k+i+1}^{j+1}\frac{1-c+p_{k+s}}{1-c-p_{k+s-1}}\right)>1. \tag{15}$$

Then every bounded solution of Eq. (1) *oscillates.*

REMARK 1. Corollary 1 improves Theorem A. In fact, when $k=1$, it is easy to verify that (15) implies (2). When $k\ge 2$, in view of Theorem A, if $\limsup_{n\to\infty} p_n\ge\frac{2}{3}(1-c)$, then every bounded solution of Eq. (1) oscillates. Therefore, we only consider the case when $p_n\le\frac{2}{3}(1-c)$.

Note that

$$\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}\frac{p_{i+j}}{1-c+p_{k+1+j}}\left(\prod_{s=j-k+i+1}^{j+1}\frac{1-c+p_{k+s}}{1-c-p_{k+s-1}}\right)=$$
$$=\frac{1}{1-c}\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}p_{j+i}\frac{1}{1-(1/(1-c))p_{j+k}}\left(\prod_{s=j-k+i+1}^{j}\frac{1-c+p_{s+k}}{1-c-p_{s+k-1}}\right)>$$
$$=\frac{1}{1-c}\sum_{i=0}^{k-1}\sum_{j=n}^{n+k-i}p_{j+i}=\frac{1}{1-c}\left(\sum_{i=0}^{k-1}(i+1)p_{n+i}+kp_{n+1}\right)>$$
$$>\frac{1}{1-c}\sum_{i=0}^{k-1}(i+1)p_{n+i}.$$

This shows that (15) also implies (2). On the other hand, we can easily cite an example to show that condition (2) does not imply condition (5). We omit it.

THEOREM 2. *Assume that* $c<0$, $k>m$ *and that*

(16) $$\limsup_{n\to\infty}\frac{p_n}{p_{n-m}}=\alpha\in(0,\infty)$$

and

(17) $$\limsup_{n\to\infty}\sum_{i=0}^{k-m-1}\sum_{j=n}^{n+k-m-i}\frac{q_i(j)}{1-c\alpha+q_{k-m+1}(j)}\times$$
$$\times\left(\prod_{s=j-(k-m)+i+1}^{j+1}\frac{1-c\alpha+q_{k-m+1}(s-1)}{1-c\alpha-q_{k-m}(s-1)}\right)>1,$$

where $q_i(n)=((i+u-2)^{(u-2)}/(u-2)!)p_{i+n}$. *Then every bounded solution of Eq.* (5) *oscillates.*

PROOF. For the sake of contradiction, assume that (5) has an bounded eventually positive solution $\{x_n\}$. Set $y_n=x_n-cx_{n-m}$. Then $\{y_n\}$ is bounded and $\Delta^u y_n=p_nx_{n-k}\geq 0$. It follows that $\Delta^i y_n$ $(i=1,2,,...,u-1)$ are monotone and each of them doesn't change sign eventually. In view of [1. Theorem 1.7.11], there exists an integer $n_1>n_0$ such that

(18) $$(-1)^i\Delta^i y_n>0\quad\text{for}\quad n\geq n_1\quad\text{and}\quad\lim_{n\to\infty}\Delta^i y_n=0,\quad i=0,\,1,2,...,u-1$$

In view of (17) there must exist a constant $\mu>1$ such that

(19) $$\limsup_{n\to\infty}\sum_{i=0}^{k-m-1}\sum_{j=n}^{n+k-m-i}\frac{q_i(j)}{1-\mu c\alpha+q_{k-m+1}(j)}\times$$
$$\times\left(\prod_{s=j-(k-m)+i+1}^{j+1}\frac{1-\mu c\alpha+q_{k-m+1}(s-1)}{1-\mu c\alpha-q_{k-m}(s-1)}\right)>1.$$

For this μ, it follows from (16) that there must exist an integer $n_2\geq n_1$ such that

(20) $$-c\frac{p_n}{p_{n-m}}\leq-\mu c\alpha,\qquad n\geq n_2$$

clearly

$$\Delta^u y_n-c\frac{p_n}{p_{n-m}}\Delta^u y_{n-m}=p_nx_{n-k}-c\frac{p_n}{p_{n-m}}p_{n-m}x_{n-m-k},$$

that is

$$\Delta^u y_n-c\frac{p_n}{p_{n-m}}\Delta^u y_{n-m}=p_ny_{n-k},\qquad n\geq n_2.$$

Substituting (20) into the obove equation, we obtain

$$\Delta^u(y_n - \mu c\alpha y_{n-m}) \geq p_n y_{n-k}. \tag{21}$$

Set $z_n = y_n - \mu c\alpha y_{n-m}$, then $\{z_n\}$ is bounded. Meanwhile, there must exist an integer $N > n_2$ such that

$$\begin{gathered} \Delta^u z_n \geq p_n y_{n-k}, \\ (-1)^i \Delta^i z_n > 0 \quad \text{for} \quad n \geq N \quad \text{and} \quad \lim_{n\to\infty} \Delta^i z_n = 0 \quad i = 0,1,2,\cdots,u-1. \end{gathered} \tag{22}$$

On the other hand, we have

$$z_n = y_n - \mu c\alpha y_{n-m} \leq y_{n-m} - \mu c\alpha y_{n-m} = (1-\mu c\alpha) y_{n-m},$$

and then

$$y_n \geq \frac{1}{1-\mu c\alpha} z_{n+m}.$$

Therefore, we obtain

$$\Delta^u z_n \geq \frac{1}{1-\mu c\alpha} p_n z_{n+m-k}. \tag{23}$$

Summing (23) from $n > N$ to ∞ for $(u-1)$ times and applying (22), we have

$$\Delta z_n + \frac{1}{1-\mu c\alpha} \sum_{i=n}^{\infty} \frac{(i-n+u-2)^{(u-2)}}{(u-2)!} p_i z_{i-(k-m)} \leq 0,$$

that is

$$\Delta z_n + \frac{1}{1-\mu c\alpha} \sum_{i=0}^{\infty} \frac{(i+u-2)^{(u-2)}}{(u-2)!} p_{n+i} z_{n+i-(k-m)} \leq 0.$$

Then we have

$$\Delta z_n + \frac{1}{1-\mu c\alpha} \sum_{i=0}^{\infty} q_i(n) z_{n+i-(k-m)} \leq 0. \tag{24}$$

It follows from (24), we have

$$\Delta z_n + \frac{1}{1-\mu c\alpha} \sum_{i=0}^{k-m+1} q_i(n) z_{n+i-(k-m)} \leq 0, \quad n \geq N, \tag{25}$$

and so

$$z_{n+1} - z_n + \frac{1}{1-\mu c\alpha} q_i(n) z_n \leq 0, \quad 0 \leq i \leq k-m.$$

Hence

$$0 < z_{n+1} \le \left(1 - \frac{1}{1-\mu c\alpha} q_i(n)\right) z_n ,$$

which implies

$$1 - \frac{1}{1-\mu c\alpha} q_i(n) > 0, \qquad 0 \le i \le k-m. \tag{26}$$

Set $w_n = z_n \prod_{i=N}^{n} \frac{1-\mu c\alpha + q_{k-m+1}(i-1)}{1-\mu c\alpha + q_{k-m+1}(i-1)}$. Similar to the proof of Theorem 1, from (25) we have

$$\Delta w_n + \frac{1}{1-\mu c\alpha + q_{k-m+1}(n)} \sum_{i=0}^{k-m-1} \left(\prod_{j=n+i-(k-m)+1}^{n+1} \frac{1-\mu c\alpha + q_{k-m+1}(j-1)}{1-\mu c\alpha - q_{k-m}(j-1)} \right) \times \tag{27}$$
$$\times\, q_i(n) w_{n+i-(k-m)} \le 0 .$$

Summing (27) from N to ∞, we obtain

$$w_N \ge \sum_{n=N}^{\infty} \frac{1}{1-\mu c\alpha + q_{k-m+1}(n)} \sum_{i=0}^{k-m-1} \left(\prod_{j=n-(k-m)+i+1}^{n+1} \frac{1-\mu c\alpha + q_{k-m+1}(j-1)}{1-\mu c\alpha - q_{k-m}(j-1)} \right) \times$$
$$\times\, q_i(n)\, w_{n+i-(k-m)} \ge$$
$$\ge \sum_{i=0}^{k-m-1} \sum_{n=N}^{N+k-m-i} \frac{1}{1-\mu c\alpha + q_{k-m+1}(n)} \left(\prod_{j=n-(k-m)+i+1}^{n+1} \frac{1-\mu c\alpha + q_{k-m+1}(j-1)}{1-\mu c\alpha - q_{k-m}(j-1)} \right) \times$$
$$\times\, q_i(n)\, w_{n+i-(k-m)} \ge$$
$$\ge w_N \sum_{i=0}^{k-m-1} \sum_{n=N}^{N+(k-m)-i} \frac{q_i(n)}{1-\mu c\alpha + q_{k-m+1}(n)} \left(\prod_{j=n-(k-m)+i+1}^{n+1} \frac{1-\mu c\alpha + q_{k-m+1}(j-1)}{1-\mu c\alpha - q_{k-m}(j-1)} \right).$$

It follows that

$$\sum_{i=0}^{k-m-1} \sum_{n=N}^{N+(k-m)-i} \frac{q_j(n)}{1-\mu c\alpha + q_{k-m+1}(n)} \times \tag{28}$$
$$\times \left(\prod_{j=n-(k-m)+i+1}^{j+1} \frac{1-\mu c\alpha + q_{k-m+1}(j-1)}{1-\mu c\alpha - q_{k-m}(j-1)} \right) \le 1.$$

Taking the limit superior as $N \to \infty$, we obtain

$$\limsup_{N\to\infty}\sum_{i=0}^{k-m-1}\sum_{n=N}^{N+k-m-i}\frac{q_i(n)}{1-\mu c\alpha+q_{k-m+1}(n)}\times$$
$$\times\left(\prod_{j=n-(k-m)+i+1}^{n+1}\frac{1-\mu c\alpha+q_{k-m+1}(j-1)}{1-\mu c\alpha-q_{k-m}(j-1)}\right)\le 1.$$

That is

$$\limsup_{N\to\infty}\sum_{i=0}^{k-m-1}\sum_{j=n}^{n+k-m-i}\frac{q_i(j)}{1-\mu c\alpha+q_{k-m+1}(j)}\times$$
$$\times\left(\prod_{s=j-(k-m)+i+1}^{j+1}\frac{1-\mu c\alpha+q_{k-m+1}(s-1)}{1-\mu c\alpha-q_{k-m}(s-1)}\right)\le 1,$$

which contradicts (19). The proof is complete.

COROLLARY 2. *Assume that* $c<0$, $k>m$ *and that*

(29) $$\limsup_{n\to\infty}\frac{p_n}{p_{n-m}}=\alpha\in(0,\infty)$$

and

(30) $$\limsup_{n\to\infty}\sum_{i=0}^{k-m-1}\sum_{j=n}^{n+k-m-i}\frac{p_{j+i}}{1-c\alpha+p_{j+k-m+1}}\times$$
$$\times\left(\prod_{s=j-k+m+i+1}^{j+1}\frac{1-c\alpha+p_{s+k-m}}{1-c\alpha-p_{s+k-m-1}}\right)>1.$$

Then every bounded solution of Eq. (1) *oscillates.*

REMARK 2. Similarly, we can prove that Corollary 2 improves Theorem B.

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