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INEQUALITIES USEFUL IN THE THEORY OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

ABSTRACT: In this paper we establish some new integral inequalities in two independent variables which can be used as tools in the theory of certain classes of hyperbolic partial differential equations. The analysis used in the proofs is elementry and the results established provide new estimates on these types of inequalities.

KEY WORDS: inequalities, partial differential equations, new estimates, non-self-adjoint, initial-boundary conditions, bounds on the solutions.

1. INTRODUCTION

In the study of qualitative behavior of solutions of partial differential equations certain integral inequalities often play a fundamential role. In [1] the author has given several integral inequalities which furnish explicit bounds on unknown functions, which can be used as handy tools in the theory of differential and integral equations. In view of the successful utilizations of such inequalities, it is natural to expect that some new integral inequalities of these types would also be equally important in order to achieve a diversity of desired goals. The aim the present paper is to establish some new integral inequalities in two independent variables which can be used as tools in the analysis of certain new and general classes of partial differential equations. We also present an immediate application to show the importance of our results to the literature.

2. STATEMENT OF RESULTS

For convenience we list the following hypotheses used in our discussion.

(H_1) $a(x,y)$ is a real-valued nonnequtive continuous function for $x,y \in R_+ = [0,\infty)$ and nondecreasing in both variables.

(H_2) $u(x,y)$, $p(x,y)$, $b(x,y)$ are real-valued nonnegative continuous functions for $x,y \in R_+$.

(H_3) $u(x,y)$, $g(x,y)$, $h(x,y)$ are real-valued nonnegative continuous functions for $x,y \in R_+$.

(H_4) $l : R_+^3 \to R_+$ be a continuous function satisfying the condition $0 \le L(x,y,v_1) - L(x,y,v_2) \le M(x,y,v_2)(v_1 - v_2)$, for $v_1 \ge v_2 \ge 0$, where $M : R_+^3 \to R_+$ is a continuous function.

Our main results are established in the following theorems.

THEOREM 1. *Suppose* (H_1), (H_2) *and* (H_4) *are true.*

(a_1) *If*

(2.1)
$$u(x,y) \le a(x,y) + p(x,y)\int_0^x b(s,y)u(s,y)ds + \int_0^x\int_0^y L(s,t,u(s,t))\,dt\,ds,$$

for $x, y \in R_+$, *then*

(2.2)
$$u(x,y) \le f(x,y)\left[a(x,y) + e(x,y)\exp\left(\int_0^x\int_0^y M(s,t,f(s,t)a(s,t)) \times f(s,t)dtds\right)\right],$$

for $x, y \in R_+$, *where*

(2.3)
$$f(x,y) = 1 + p(x,y)\int_0^x b(s,y)\exp\left(\int_s^x b(\sigma,y)p(\sigma,y)d\sigma\right)ds,$$

(2.4)
$$e(x,y) = \int_0^x\int_0^y L(s,t,f(s,t)a(s,t)\,dt\,ds,$$

for $x, y \in R_+$.

(a_2) *If*

(2.5)
$$u(x,y) \le a(x,y) + p(x,y)\int_0^y b(x,t)u(x,t)dt + \int_0^x\int_0^y L(s,t,u(s,t))\,dt\,ds,$$

for $x, y \in R_+$, *then*

(2.6)
$$u(x,y) \le \bar{f}(x,y)\left[a(x,y) + \bar{e}(x,y)\exp\left(\int_0^x\int_0^y M(s,t,\bar{f}(s,t)a(s,t)) \times \bar{f}(s,t)dtds\right)\right],$$

for $x, y \in R_+$, *where*

(2.7)
$$\bar{f}(x,y) = 1 + p(x,y)\int_0^y b(x,t)\exp\left(\int_t^y b(x,\tau)p(x,\tau)\,d\tau\right)dt,$$

(2.8)
$$\bar{e}(x,y) = \int_0^x\int_0^y L(s,t,\bar{f}(s,t)\,a(s,t)\,dt\,ds,$$

for $x, y \in R_+$.

THEOREM 2. *Suppose* (H_1), (H_2) *and* (H_4) *are true.*

(b_1) *If*

(2.9)
$$u(x,y) \le a(x,y) + \int_0^x g(s,y)\left(u(s,y) + \int_0^s h(\sigma,y)u(\sigma,y)\,d\sigma\right)ds + \int_0^x\int_0^y L(s,t,u(s,t))\,dt\,ds,$$

for $x, y \in R_+$, *then*

(2.10)
$$u(x,y) \le k(x,y)\left[a(x,y) + E(x,y)\exp\left(\int_0^x\int_0^y M(s,t,k(s,t)\,a(s,t)) \times k(s,t)dtds\right)\right],$$

for $x, y \in R_+$, *where*

(2.11)
$$k(x,y) = 1 + \int_0^x g(s,y)\exp\left(\int_0^s [g(\sigma,y) + h(\sigma,y)]d\sigma\right)ds,$$

(2.12)
$$E(x,y) = \int_0^x\int_0^y L(s,t,k(s,t)\,a(s,t)\,dt\,ds,$$

for $x, y \in R_+$.

(b_2) *If*

$$u(x,y) \le a(x,y) + \int_0^x g(x,t)\left[u(x,t) + \int_0^t h(x,\tau)u(x,\tau)d\tau\right]dt + \tag{2.13}$$
$$+ \int_0^x\int_0^y L(s,t,u(s,t))\,dt\,ds,$$

for $x, y \in R_+$, *then*

$$u(x,y) \le \bar{k}(x,y)\left[a(x,y) + \bar{E}(x,y)\exp\left(\int_0^x\int_0^y M(s,t,\bar{k}(s,t)\,a(s,t))\times\right.\right. \tag{2.14}$$
$$\left.\left.\times\,\bar{k}(s,t)\,dt\,ds\right)\right],$$

for $x, y \in R_+$, *where*

$$\bar{k}(x,y) = 1 + \int_0^y g(x,t)\exp\left(\int_0^t [g(x,\tau) + h(x,\tau)]d\tau\right)dt, \tag{2.15}$$

$$\bar{E}(x,y) = \int_0^x\int_0^y L(s,t,\bar{k}(s,t)\,a(s,t))\,dt\,ds, \tag{2.16}$$

for $x, y \in R_+$.

3. PROOFS THEOREMS 1 AND 2

We give the details of the proofs of (a_1) and (b_1) only. The proofs of (a_2) and (b_2) can be completed by following the proofs of the above mentioned inequalities with suitable modifications.

(a_1) Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + \int_0^x\int_0^y L(s,t,u(s,t))\,dt\,ds, \tag{3.1}$$

then (2.1) can be stated as

$$u(x,y) \le z(x,y) + p(x,y)\int_0^x b(s,y)\,u(s,y)\,ds. \tag{3.2}$$

Clearly $z(x, y)$ is nonnegative and nondecreasing function for $x, y \in R_+$. Treating (3.2) as one-dimensional integral inequality for any fixed $y \in R_+$ and a suitable application of the inequality given in Theorem 1.3.3 in [1, p. 15] yields

$$u(x, y) \leq z(x, y) f(x, y), \tag{3.3}$$

where $f(x, y)$ is defined by (2.3). From (3.1) and (3.3) we have

$$u(x, y) \leq f(x, y)[a(x, y) + r(x, y)], \tag{3.4}$$

where

$$r(x, y) = \int_0^x \int_0^y L(s, t, u(s, t))\, dt\, ds. \tag{3.5}$$

From (3.4), (3.5) and hypothesis (H_4) we observe that

$$\begin{aligned} r(x, y) \leq & \int_0^x \int_0^y \{L(s, t, f(s,t)[a(s,t) + r(s,t)]) - \\ & - L(s, t, f(s,t)\, a(s,t)) + L(s, t, f(s,t)\, a(s,t))\}\, dt\, ds \leq \\ & \leq e(x, y) + \int_0^x \int_0^y M(s, t, f(s,t)\, a(s,t)) f(s,t)\, r(s,t)\, dt\, ds, \end{aligned} \tag{3.6}$$

where $e(x, y)$ is defined by (2.4). It is easy to obsrve that $e(x, y)$ is nonnegative and nondecreasing in each variable $x, y \in R_+$. A suitable application of Theorem 4.2.2 given in [1, p. 325] yields

$$r(x, y) \leq e(x, y) \exp\left(\int_0^x \int_0^y M(s, t, f(s,t) a(s,t)) f(s,t)\, dt\, ds \right). \tag{3.7}$$

Using (3.7) in (3.4) we get the required inequality in (2.2).

(b_1) Define a function $z(x, y)$ by (3.1), then (2.9) can be written as

$$u(x, y) \leq z(x, y) + \int_0^x g(s, y) \left(u(s, y) + \int_0^s h(\sigma, y) u(\sigma, y)\, d\sigma \right) ds. \tag{3.8}$$

Clearly $z(x, y)$ is nonnegative and nondecreasing function for $x, y \in R_+$. Treating (3.8) as an one-dimensional integral inequality for any fixed $y \in R_+$ and a suitable application of Theorem 1.7.4 given in [1, p. 39] yields

$$u(x, y) \leq z(x, y) k(x, y), \tag{3.9}$$

where $k(x,y)$ is defined by (2.11). Now by following the proof of (a_1) with suitable modifications we get the desired inequality in (2.10).

4. AN APPLICATION

In this section, we indicate an application to obtain bound on the solution of a certain non-self-adjoint hyperbolic partial differential equation for which the inequalities available in the following non-self-adjoint hyperbolic partial differential equation

$$u_{xy}(x,y) = (p(x,y)u(x,y))_y + F(x,y,u(x,y)), \tag{4.1}$$

with the given initial-boundary conditions

$$u(x,0) = \alpha(x), \quad u(0,y) = \beta(y), \quad \alpha(0) = \beta(0) = 0, \tag{4.2}$$

where all the functions involved in (4.1), (4.2) are real-valued and defined on their respective domains of their definitions. It is easy to observe that the problem (4.1), (4.2) is equivalent to the following integral equation

$$u(x,y) = q(x,y) + \int_0^x p(s,y)u(s,y)\,ds + \int_0^x\int_0^y F(s,t,u(s,t))\,dt\,ds, \tag{4.3}$$

where

$$q(x,y) = \alpha(x) + \beta(y) - \int_0^x p(s,0)\alpha(s)\,ds. \tag{4.4}$$

We assume that

$$|F(x,y,u)| \le L(x,y,|u|), \tag{4.5}$$

$$|q(x,y)| \le a(x,y), \tag{4.6}$$

where $L(x,y,r)$ and $a(x,y)$ are as given in (H_4) and (H_1) respectively. From (4.3), (4.5), (4.6) we have

$$|u(x,y)| \le a(x,y) + \int_0^x |p(s,y)||u(s,y)|\,ds + \int_0^x\int_0^y L(s,t,|u(s,t)|)\,dt\,ds.$$

Now an application of the inequality given in Theorem 1 part (a_1) yields

$$|u(x,y)| \leq f_0(x,y)\Bigg[a(x,y)+e_0(x,y)\times \tag{4.7}$$

$$+\exp\left(\int_0^x\int_0^y M(s,t,f_0(s,t)\,a(s,t))f_0(s,t)\,dt\,ds\right)\Bigg],$$

where

$$f_0(x,y) = 1 + \int_0^x |p(s,y)|\exp\left(\int_s^x |p(\sigma,y)|\,d\sigma\right)ds, \tag{4.8}$$

$$e_0(x,y) = \int_0^x\int_0^y L(s,t,f_0(s,t)\,a(s,t))\,dt\,ds, \tag{4.9}$$

for $x, y \in R_+$ and the function $M(x,y,r)$ is defined as in (H_4). The right hand side of (4.7) gives the bound on the solution of problem (4.1), (4.2) in terms of the known functions.

We note that the inequality given in Theorem 1 part (a_2) can be used to obtain the bound on the solution of the non-self-adjoint hyperbolic partial differential equation

$$u_{xy}(x,y) = (p(x,y)u(x,y))_x + F(x,y,u(x,y)), \tag{4.1}$$

with the initial-boundary conditions given in (4.2), under some suitable conditions on the functions involved in (4.10) and (4.2). Further it is to be noted that the inequalities established in Theorem 2 can be used to obtain bounds on the solutions of the following non-self-adjoint hyperbolic partial integrodifferential equations

$$u_{xy}(x,y) = \left(Q_1\left(x,y,\int_0^x k_1(\sigma,y,u(\sigma,y))\,d\sigma\right)\right)_y + F(x,y,u(x,y)), \tag{4.11}$$

$$u_{xy}(x,y) = \left(Q_2\left(x,y,\int_0^y k_2(x,y,u(x,\tau))\,d\tau\right)\right)_x + F(x,y,u(x,y)), \tag{4.12}$$

with the given initial-boundary conditions in (4.2), under some suitable conditions on the functions involved in (4.11), (4.12) and (4.2). For some other applications of the above type of inequalities, see [1, 2, 3].

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