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UNIQUENESS CRITERION OF A CLASSICAL SOLUTION OF A BIPARABOLIC PROBLEM IN AN UNBOUNDED DOMAIN WITH A CURVILINEAR BOUNDARY

ABSTRACT: The aim of the paper is to prove a uniqueness criterion of a classical solution of a biparabolic problem in an unbounded domain with a curvilinear boundary.

KEY WORDS: biparabolic problem, uniqueness of a solution, classical solution, unbounded domain.

1. INTRODUCTION

In this paper we prove a theorem (Theorem 1) on the uniqueness of a classical solution of a biparabolic problem in an unbounded domain, contained in R^2, with a curvilinear boundary. For this purpose some calculation methods of the classical mathematical analysis are applied.

In papers [1], [2], [3] and [5] biparabolic problems with boundary conditions of Lauricella type for the half-plane, for the curvilinear trapezium, for the strip and for half-space, respectively, are considered. In [13] a construction of the first non-linear boundary value problem for the one-dimensional bicaloric equation in curvilinear trapezium is given. In papers [6], [7] and [8] similar uniqueness problems for the strip, for the curvilinear trapezium and for half-unbounded domain, respectively, are considered. In paper [10], a construction of the biparabolic limit problem (1) – (4) is given. In [4], [11], [12] and [13] biparabolic problems with another boundary conditions are considered.

In [9], M. Krzyżański considered the uniqueness of classical solutions of a biharmonic problem and of a initial-boundary problem for the equation of a vibrating beam.

2. PRELIMINARIES

The notation, definitions and assumptions, given in this section, are valid throughout the paper.

By T we denote a fixed positive number.

Let $p \in C^3([0,T],R)$ be a strictly decreasing function on the interval $[0,T]$.

By D we denote the following set:

$$D := \{(x,t) \in R^2 : x > p(t),\ 0 < t < T\}.$$

The above set is said to be *a domain of type (PC)*.

Given the functions $f : D \to R$, $f_i : (p(0), \infty) \to R$, $h_i : (0,T) \to R$ and $k_i : (0,T) \to R$ $(i = 0,1)$ *the Fourier's second biparabolic problem of type (PC)* consists in finding a function $u \in C^{4,2}(\overline{D}, R)$, for which the derivatives $D_x^{\alpha} D_t^{\beta} u\,(0 \le \alpha \le 4, 0 \le \beta \le 2, \alpha + 2\beta \le 4)$ are bounded in D, satisfying the equation

(1) $$P^2 u(x,t) = f(x,t) \quad \text{for} \quad (x,t) \in D,$$

where

$$P := D_x^2 - D_t,$$

satisfying the initial conditions

(2) $$D_t^i u(x,0) = f_i(x) \quad \text{for} \quad x \in (p(0), \infty) \quad (i = 0,1)$$

and satisfying the boundary conditions

(3) $$D_x^i u(p(t),t) = h_i(t) \quad \text{for} \quad t \in (0,T) \quad (i = 0,1)$$

and

(4) $$D_x^i u(\infty,t) = k_i(t) \quad \text{for} \quad t \in (0,T) \quad (i = 0,1).$$

A function u possessing the above properties is said to be *a classical solution* in D of the Fourier's second biparabolic problem of type (PC).

3. UNIQUENESS CRITERION

THEOREM 1. *Suppose that D is a set of type (PC). Then in the class of all functions $u \in C^{4,2}(\overline{D}, R)$, for which the derivatives $D_x^{\alpha} D_t^{\beta} u\,(0 \le \alpha \le 4, 0 \le \beta \le 2, \alpha + 2\beta \le 4)$ are bounded in D, the Fourier's second biparabolic problem of type (PC) admits at most one classical solution in D.*

PROOF. Suppose that $u_i \in C^{4,2}(\overline{D}, R)$ $(i = 1,2)$, for which the derivatives $D_x^{\alpha} D_t^{\beta} u_i\,(i = 1,2;\ 0 \le \alpha \le 4, 0 \le \beta \le 2, \alpha + 2\beta \le 4)$ are bounded in D, are classical solutions of problem (1) – (4).

Let

(5) $$v(y,s) := u_1(y,s) - u_2(y,s) \quad \text{for} \quad (y,s) \in \overline{D}.$$

It is easy to see that the following formulas hold:

(6) $$D_y^4 v(y,s) - 2D_y^2 D_s v(y,s) + D_s^2 v(y,s) = 0 \quad \text{for} \quad (y,s) \in \overline{D}$$

(7) $$D_y^i v(p(s),s)=0 \quad \text{for} \quad s\in[0,T] \quad (i=0,1),$$

(8) $$D_y^i v(\infty,s)=0 \quad \text{for} \quad s\in[0,T] \quad (i=0,1)$$

and

(9) $$D_s^i v(y,0)=0 \quad \text{for} \quad y\geq p(0) \quad (i=0,1).$$

Let

$$D(x,t) := \{(y,s): y\in[p(s),x],\ s\in[0,t]\}, \quad (x,s)\in D.$$

Multiplying equation (6) by $D_s v(y,s)$ and integrating over $D(x,t)$, we obtain

(10) $$I(x,t) := \int_0^t \int_{p(s)}^x D_s v(y,s)\cdot[D_y^4 v(y,s) - 2D_y^2 D_s v(y,s) + + D_s^2 v(y,s)]dyds = 0 \quad \text{for} \quad (x,t)\in D.$$

Observe that

(11) $$I(x,t) = \sum_{i=1}^{3} I_i(x,t) \quad \text{for} \quad (x,t)\in D,$$

where

(12) $$I_1(x,t) := -2\int_0^t \int_{p(s)}^x D_s v(y,s)\cdot D_y^2 D_s v(y,s)dyds \quad \text{for} \quad (x,t)\in D,$$

(13) $$I_2(x,t) := \int_0^t \int_{p(s)}^x D_s^2 v(y,s)\cdot D_s v(y,s)dyds \quad \text{for} \quad (x,t)\in D$$

and

(14) $$I_3(x,t) := \int_0^t \int_{p(s)}^x D_y^4 v(y,s)\cdot D_s v(y,s)dyds \quad \text{for} \quad (x,t)\in D.$$

It is easy to see that

(15) $$D_s^2 v(y,s)\cdot D_s v(y,s) = \frac{1}{2} D_s[D_s v(y,s)]^2 \quad \text{for} \quad (y,s)\in \overline{D}$$

and

(16) $$D_y^4 v(y,s)\cdot D_s v(y,s) = D_y[D_y^3 v(y,s)\cdot D_s v(y,s) - - D_y^2 v(y,s)\cdot D_y D_s v(y,s)] + \frac{1}{2} D_s[D_y^2 v(y,s)] \quad \text{for} \quad (y,s)\in \overline{D}.$$

Integrating by parts $I_1(x,t)$ we have that

$$(17)\quad I_1(x,t) = -2\int_0^t [D_s v(y,s)\cdot D_y D_s v(y,s)]_{y=p(s)}^{y=x} ds +$$

$$+ 2\int_0^t \int_{p(s)}^x [D_y D_s v(y,s)]^2 dyds =$$

$$= -2\int_0^t [D_s v(x,s)\cdot D_y D_s v(x,s)]ds +$$

$$+ 2\int_0^t [D_s v(p(s),s)\cdot D_y D_s v(p(s),s)]ds +$$

$$+ 2\int_0^t \int_{p(s)}^x [D_y D_s v(y,s)]^2 dyds =$$

$$= I_1^1(x,t) + I_1^2(t) + I_1^3(x,t) \quad \text{for} \quad (x,t)\in D,$$

where

$$(18)\quad I_1^1(x,t) := -2\int_0^t [D_s v(x,s)\cdot D_y D_s v(x,s)]ds \quad \text{for} \quad (x,t)\in D,$$

$$(19)\quad I_1^2(t) := 2\int_0^t [D_s v(p(s),s)\cdot D_y D_s v(p(s),s)]ds \quad \text{for} \quad t\in (0,T)$$

and

$$(20)\quad I_1^3(x,t) := 2\int_0^t \int_{p(s)}^x [D_y D_s v(y,s)]^2 dyds \quad \text{for} \quad (x,t)\in D.$$

For I_1^1 we have the following estimation:

$$|I_1^1(x,t)| \le 2\left(\sup_{(y,s)\in D} |D_s v(y,s)|\right)\cdot\left|\int_0^t D_s D_y v(x,s)ds\right| =$$

$$= 2\left(\sup_{(y,s)\in D} |D_s v(y,s)|\right)\cdot|D_y v(x,t) - D_y v(x,0)| \quad \text{for} \quad (x,t)\in D.$$

From the above estimation and from condition (8) (where $i = 1$), we obtain that

$$(21)\quad \lim_{x\to\infty} |I_1^1(x,t)| \le 2\left(\sup_{(y,s)\in D} |D_s v(y,s)|\right)\cdot|D_y v(\infty,t) -$$

$$- D_y v(\infty,0)| = 0 \quad \text{for} \quad t\in (0,T).$$

By (19) and (7),

$$(22)\quad |I_1^2(t)| \le 2\left(\sup_{s\in[0,T]} |D_s v(p(s),s)|\right)\cdot\left|\int_0^t D_y D_s v(p(s),s)ds\right| =$$

$$= 2\left(\sup_{s\in[0,T]} |D_s v(p(s),s)|\right)\cdot\left|\int_0^t D_s D_y v(p(s),s)ds\right| =$$

$$= 2\left(\sup_{s\in[0,T]} |D_s v(p(s),s)|\right)\cdot |D_y v(p(t),t) - D_y v(p(0),0)| = 0$$

for $t \in [0,T]$.

Consequently, from (17), (21) and (20), we have the formula

$$(23)\qquad \lim_{x\to\infty} I_1(x,t) = 2\int_0^t \int_{p(s)}^x [D_y D_s v(y,s)]^2 dyds \ge 0 \quad \text{for } t\in(0,T).$$

Formulas (13) and (15), the assumption that the function p is decreasing on $[0,T]$ and formula (9) (where $i = 1$) imply that

$$I_2(x,t) = \frac{1}{2}\int_0^t \int_{p(s)}^x D_s[D_s v(y,s)]^2 dyds \ge \frac{1}{2}\int_{p(0)}^x \left[\int_0^t D_s[D_s v(y,s)]^2 ds\right]dy =$$

$$= \frac{1}{2}\int_{p(0)}^x \left([D_s v(y,s)]^2\big|_{s=t} - [D_s v(y,s)]^2\big|_{s=0}\right)dy =$$

$$= \frac{1}{2}\int_{p(0)}^x [D_s v(y,t)]^2 dy \ge 0 \quad \text{for } (x,t)\in D.$$

Therefore,

$$(24)\qquad \lim_{x\to\infty} I_2(x,t) \ge \frac{1}{2}\int_{p(0)}^\infty [D_s v(y,t)]^2 dy \ge 0 \quad \text{for } t\in(0,T).$$

Next, from (14) and (16), and from the fact that the function p is decreasing on $[0,T]$, we have that

$$(25)\quad I_3(x,t) = \int_0^t \int_{p(s)}^x D_y[D_y^3 v(y,s)\cdot D_s v(y,s) - D_y^2 v(y,s)\cdot D_y D_s v(y,s)]dyds +$$

$$+ \frac{1}{2}\int_0^t \int_{p(s)}^x D_s[D_y^2 v(y,s)]^2 dyds \ge$$

$$\ge \int_0^t [D_y^3 v(y,s)\cdot D_s v(y,s) - D_y^2 v(y,s)\cdot D_y D_s v(y,s)]_{y=p(s)}^{y=x} ds +$$

$$+\frac{1}{2}\int_{p(0)}^{x}\int_{0}^{t} D_s[D_y^2 v(y,s)]^2 ds dy =$$

$$=\int_{0}^{t}[D_y^3 v(x,s)\cdot D_s v(x,s)-D_y^2 v(x,s)\cdot D_y D_s v(x,s)-$$

$$-D_y^3 v(p(s),s)\cdot D_s v(p(s),s)+D_y^2 v(p(s),s)\cdot D_y D_s v(p(s),s)]ds+$$

$$+\frac{1}{2}\int_{p(0)}^{x}[D_y^2 v(y,s)]^2\Big|_{s=0}^{s=t} dy =$$

$$=\sum_{i=1}^{2} I_3^i(x,t)+\sum_{i=3}^{4} I_3^i(t)+I_3^5(x,t) \quad \text{for} \quad (x,t)\in D,$$

where

$$I_3^1(x,t):=\int_0^t D_y^3 v(x,s)\cdot D_s v(x,s)ds \quad \text{for } (x,t)\in D,$$

$$I_3^2(x,t):=-\int_0^t D_y^2 v(x,s)\cdot D_y D_s v(x,s)ds \quad \text{for } (x,t)\in D,$$

$$I_3^3(t):=-\int_0^t D_y^3 v(p(s),s)\cdot D_s v(p(s),s)ds \quad \text{for } t\in(0,T),$$

$$I_3^4(t):=\int_0^t D_y^2 v(p(s),s)\cdot D_y D_s v(p(s),s)ds \quad \text{for } t\in(0,T)$$

and

$$I_3^5(x,t):=\frac{1}{2}\int_{p(0)}^{x}\left([D_y^2 v(y,t)]^2-[D_y^2 v(y,0)]^2\right)dy \quad \text{for } (x,t)\in D.$$

For I_3^1 we have the following estimation:

$$|I_3^1(x,t)|\le\left(\sup_{(y,s)\in D}|D_y^3 v(y,s)|\right)\cdot\left|\int_0^t D_s v(x,s)ds\right|=$$

$$=\left(\sup_{(y,s)\in D}|D_y^3 v(y,s)|\right)\cdot|v(x,t)-v(x,0)| \quad \text{for } (x,t)\in D.$$

By the above formula and by condition (8) (where $i=0$), we get

$$\lim_{x\to\infty}|I_3^1(x,t)|\le\left(\sup_{(y,s)\in D}|D_y^3 v(y,s)|\right)\cdot|v(\infty,t)-v(\infty,0)|=0 \tag{26}$$

$$\text{for } t\in(0,T).$$

Integrating I_3^2 by parts, we have

$$I_3^2(x,t) = -\int_0^t D_y^2 v(x,s)\cdot D_y D_s v(x,s)ds =$$

$$= - D_y^2 v(x,s)\cdot D_y v(x,s)|_{s=0}^{s=t} + \int_0^t D_s D_y^2 v(x,s)\cdot D_y v(x,s)ds =$$

$$= - D_y^2 v(x,t)\cdot D_y v(x,t) + D_y^2 v(x,0)\cdot D_y v(x,0) +$$

$$+ \int_0^t D_s D_y^2 v(x,s)\cdot D_y v(x,s)ds \quad \text{for } (x,t)\in D.$$

Consequently, from (8) (where $i = 1$)

$$\text{(27)}\quad \lim_{x\to\infty} | I_3^2(x,t) | \le \left(\sup_{(y,s)\in D} | D_y^2 v(y,s) |\right)\cdot | D_y v(\infty,t) | + $$

$$+ \left(\sup_{(y,0)\in \overline{D}} | D_y^2 v(y,0) |\right)\cdot | D_y v(\infty,0) | +$$

$$+ \left(\sup_{(y,s)\in D} | D_s D_y^2 v(y,s) |\right)\cdot \left|\int_0^t D_y v(\infty,s)ds\right| = 0 \quad \text{for } (x,t)\in D.$$

Arguing similarly as for I_3^1 and I_3^2, we obtain the following formulas:

$$\text{(28)}\quad | I_3^3(t) | \le \left(\sup_{s\in[0,T]} | D_y^3 v(p(s),s) |\right)\cdot \left|\int_0^t D_s v(p(s),s)ds\right| =$$

$$= \left(\sup_{s\in[0,T]} | D_y^3 v(p(s),s) |\right)\cdot | v(p(t),t) - v(p(0),0) | = 0$$

for $t\in (0,T)$

and

$$\text{(29)}\quad | I_3^4(t) | \le \left(\sup_{s\in[0,T]} | D_y^2 v(p(s),s) |\right)\cdot \left|\int_0^t D_s D_y v(p(s),s)ds\right| =$$

$$= \left(\sup_{s\in[0,T]} | D_y^2 v(p(s),s) |\right)\cdot | D_y v(p(t),t) - D_y v(p(0),0) | = 0$$

for $t\in (0,T)$.

Observe that

$$I_3^5(x,t) = \frac{1}{2}\int_{p(0)}^x [D_y^2 v(y,t)]^2 dy - \frac{1}{2}\int_{p(0)}^x [D_y^2 v(y,0)]^2 dy \quad \text{for } (x,t)\in D.$$

Since

$$\int_{p(0)}^{x} [D_y^2 v(y,0)]^2 dy \le \left(\sup_{(y,s)\in D} |D_y^2 v(y,s)| \right) \cdot \left| \int_{p(0)}^{t} D_y^2 v(y,0) dy \right| =$$

$$= \left(\sup_{(y,s)\in D} |D_y^2 v(y,s)| \right) \cdot | D_y v(x,0) - D_y v(p(0),0) |$$

for $x > p(0)$, then, by (7) and (8),

$$\lim_{x\to\infty} \int_{p(0)}^{x} [D_y^2 v(y,0)]^2 dy \le \left(\sup_{(y,s)\in D} |D_y^2 v(y,s)| \right) \cdot | D_y v(\infty,0) - D_y v(p(0),0) | = 0.$$

Therefore,

(30) $$\lim_{x\to\infty} I_3^5(x,t) = \frac{1}{2} \int_{p(0)}^{x} [\mathrm{D}_y^2 v(y,t)]^2 dy \ge 0 \quad \text{for } t \in (0,T).$$

Consequently, from (25)-(30),

(31) $$\lim_{x\to\infty} I_3(x,t) \ge \frac{1}{2} \int_{p(0)}^{x} [\mathrm{D}_y^2 v(y,t)]^2 dy \ge 0 \quad \text{for } t \in (0,T).$$

By (10),

(32) $$\lim_{x\to\infty} I(x,t) = 0 \quad \text{for } t \in (0,T).$$

Formulas (32), (11), (23), (24) and (31) imply that

$$0 = \lim_{x\to\infty} I(x,t) \ge 2 \int_0^t \int_{p(s)}^{\infty} [D_y D_s v(y,s)]^2 dy ds +$$

$$+ \frac{1}{2} \int_{p(0)}^{\infty} [D_s v(y,t)]^2 dy + \frac{1}{2} \int_{p(0)}^{\infty} [D_y^2 v(y,t)]^2 dy \ge 0 \quad \text{for } t \in (0,T).$$

From the above formula we have, particularly, that

(33) $$D_y D_s v(y,s) = 0 \quad \text{for } y \ge p(s), \quad 0 \le s \le t \le T$$

and

(34) $$D_s v(y,s) = 0 \quad \text{for } y \ge p(0), \quad 0 \le t \le T.$$

By (33) and (34), we obtain that

(35) $$D_s v(y,s) = D_s v(p(0),0) = 0 \quad \text{for } y \ge p(s), \quad 0 \le s \le t \le T.$$

Consequently, by (35) and (9),

$$v(y,s) = v(y,0) = 0 \quad \text{for} \quad (y,s) \in \overline{D}.$$

It means that

$$u_1 \equiv u_2 \quad \text{in} \quad \overline{D}.$$

The proof of Theorem 1 is complete.

4. REMARK

It is easy to see that if, in the class of all functions $u \in C^{4,2}(\overline{D}, R)$, for which the derivatives $D_x^\alpha D_t^\beta u \, (0 \le \alpha \le 4, 0 \le \alpha \le 2, \alpha + 2\beta \le 4)$, are bounded in D, there exists the unique classical solution in D of the Fourier's second biparabolic problem of type (PC) then the functions f, f_i, h_i and k_i $(i=0,1)$ must be defined, respectively, in the sets $\overline{D}$, $[p(0), \infty)$ and $[0,T]$. Moreover, the above functions must be suitably regular in the above sets and they must satisfy the compatibility conditions.

Results of publication [8] imply that the biparabolic problem (1)-(4), where $k_i \equiv 0$ $(i=0,1)$, has a classical solution in D. However, the results from [8] do not imply the uniqueness of the classical solution in D of the Fourier's second biparabolic problem of type (PC). It is the reason that, in this paper, we study the existence of a classical solution in D of the above problem.

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