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APPROXIMATION OF ENTIRE HARMONIC FUNCTIONS IN R^3 IN L^β -NORM

ABSTRACT: A function which is harmonic in a neighbourhood of the origin in R^3 has there an expansion in spherical harmonics. For H_R, he class of all harmonic functions regular in $D_R=\{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2<R^3\}$ and continuous on $\overline{D}_R$, the closure of D_R, we define approximation error as $E_n^\beta(H,R)=\inf_{g\in\pi_n}\|H-g\|_{\beta,R}$, $n=0,1,...,\pi_n$ is the set of all harmonic polynomials of degree at most n. In this paper we have obtained the necessary and sufficient condition on the rate of decrease of $E_n^\beta(H,R)$ such that $H\in H_R$ has an analytic continuation as an entire harmonic function with (p,q)-growth.

KEY WORDS: harmonic functions, index-pair, proximate order, (p,q)-type, approximation error, L^β-norm.

1. INTRODUCTION

The harmonic functions in R^3 are the solutions of the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2}+\frac{\partial^2 H}{\partial x_2^2}+\frac{\partial^2 H}{\partial x_3^2}=0. \tag{1.1}$$

A harmonic function regular about the origin ean be expanded as

$$H\equiv H(r,\theta,\phi)=\sum_{n=0}^{\infty}r^n\sum_{m=0}^{n}\left(a_{nm}^{(1)}\cos m\phi+a_{nm}^{(2)}\sin m\phi\right)P_n^m(\cos\theta), \tag{1.2}$$

where $x_1=r\cos\theta$, $x_2=r\sin\theta\cos\phi$, $x_3=r\sin\theta\sin\phi$ and $P_n^m(t)$ are associated Legendre's functions of the first kind of degree m and order n. A harmonic polynomial of degree k is a polynomial of degree k in x_1, x_2 and x_3 which satisfies (1.1).

A harmonic function is said to be regular in $D_R=\{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2<R^3, 0<R\le\infty\}$ if the series (1.2) converges uniformly on every compact subset of D_R. A harmonic function is called entire if it is regular in D_∞. The concepts of *index-pair* (p,q), $p\ge q\ge 1$, (p,q)*-order and* (p,q)*-type* were introduced by Juneja *et al.* ([5], [6]).

The $(p,q)-$order of an entire harmonic function is defined as

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(r,H)}{\log^{[q]} r} = \rho(p,q) \equiv \rho,$$

and, the function having $(p,q)-$order ρ $(b<\rho(p,q)<\infty)$ is said to be of $(p,q)-$type T if

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,H)}{(\log^{[q-1]} r)^{\rho}} = T(p,q) \equiv T, \tag{1.3}$$

where $M(r,H)=\max_{x_1^2+x_2^2+x_3^2=r^2} |H(x_1,x_2,x_3)|$ $b=1$ if $p=q$ and $b=0$ otherwise.

The growth parameters ρ and T of an entire harmonic function with index-pair $(p,q)=(2,1)$ have been studied in L^2-norm by Fryant [4] but these concepts are inadequate to compare the growth of those entire functions which are of the same order but of infinite type. Hence, for a refinement of the above scale one may utilize the concept of proximate order cf., ([7], [1]).

A positive function $\rho(r)$ defined on $[r_0,\infty)$, $r_0>\exp^{[q-1]}1$, is said to be a proximate order of an entire function with index-pair (p,q) if

(*i*) $\rho(r)\to\rho(p,q)\equiv\rho$ as $r\to\infty$, $b<\rho<\infty$;

(*ii*) $\Lambda_{[q]}(r)\rho'(r)\to 0$ as $r\to\infty$,

where $\rho'(r)$ denotes the derivative of $\rho(r)$, and, for convenience, $\Lambda_{[q]}(r)=\prod_{i=0}^{q}\log^{[i]} r$.

We now define (p,q)-type T^* of H with respect to a given proximate order $\rho(r)$ as

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,H)}{(\log^{[q-1]} r)^{\rho(r)}} = T^*(p,q) \equiv T^*.$$

If the quantity T^* is different from zero and infinity then $\rho(r)$ is said to be the proximate order of a given entire harmonic function H with index-pair (p,q).

For H_R, $0<R<\infty$, the class of all harmonic functions regular in D_R and continuous on $\overline{D}_R$ (the closure of the open disc D_R), set

$$\| H \|_{\beta,R} = \left[\iint_{\overline{D}_R} |H|^{\beta}\, dxdy \right]^{\frac{1}{\beta}}, \qquad 1\le\beta<\infty,$$

where $\|\cdot\|_{\beta,R}$ denotes L^{β}-norm. The approximation error is defined as

$$E_n^\beta(H,R) = \inf_{g\in\pi_n} \| H - g \|_{\beta,R}, \qquad n = 0,1,\dots,$$

where π_n denotes the set of all polynomials of degree at most *n*. For ($\beta = \infty$, the above norm is understood to be the sup norm:

$$\| H - g \|_{R,\infty} = \max_{(x_1,x_2,x_3)\in D_R} | H(x_1,x_2,x_3) - g(x_1,x_2,x_3) |.$$

In this paper we have obtained the necessary and sufficient condition on the rate of decrease of $E_n^\beta(H,R)$ such that $H \in H_R$ has analytic continuation as an entire harmonic function with (p,q)-growth.

2. Auxiliary Results

Here, we mention some results, which will be utilized in the sequel. The function $(\log^{[q-1]} r)^{\rho(r)-A}$ is a monotonically increasing function of *r* for $r > r_0$. This can be verified just by noting the fact that its first derivative is positive for $r > r_0$. This ensures to define $F(x)$ to be the unique solution of the equation

$$x = (\log^{[q-1]} r)^{\rho(r)-A} \Leftrightarrow F(x) = \log^{[q-1]} r,$$

where $A = 1$, if $(p,q) = (2,2)$ and $A = 0$, otherwise.

By the definition of proximate order and using the above equation, we have

$$\frac{d[\log F(x)]}{d[\log x]} = \frac{1}{\rho(r) - A + \Lambda_{[q]}(r)\rho'(r)} \Leftrightarrow \lim_{x\to\infty} \frac{d[\log F(x)]}{d[\log x]} = \frac{1}{\rho - A}.$$

Thus, for given $\varepsilon > 0$ and $x > x_0$, it follows that for $\eta > 1$

$$\int_x^{\eta x} \left(\frac{1}{\rho - A} - \varepsilon \right) d[\log t] < \int_x^{\eta x} d[\log F(t)] < \int_x^{\eta x} \left(\frac{1}{\rho - A} + \varepsilon \right) d[\log t]$$

or

$$\left(\frac{1}{\rho - A} - \varepsilon \right) \log\eta < \log \frac{F(\eta x)}{F(x)} < \left(\frac{1}{\rho - A} + \varepsilon \right) \log\eta$$

or

$$\eta^{-\varepsilon + 1/(\rho - A)} < \frac{F(\eta x)}{F(x)} < \eta^{\varepsilon + 1/(\rho - A)}.$$

Consequently,

$$\lim_{x\to\infty} \frac{F(\eta x)}{F(x)} = \eta^{1/\rho - A} \quad \text{uniformly for every } \eta,\ 1 < \eta < \infty.$$

The inclusion of $0 < \eta \le 1$ is simple, and hence left for the reader.

LEMMA 1. *Let $H \in H_R$ be an entire harmonic function of (p,q)-order p. Then*

$$\rho(p,q) = P(L(p,q)),$$

where

$$L(p,q) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q]} \alpha_n^{-1/n}},$$

$$P(L(p,q)) = \begin{cases} L(p,q) & \text{if } q<p<\infty, \\ 1+L(p,q) & \text{if } p=q=2, \\ \max(1, L(p,q)) & \text{if } 3\le \mathrm{p}=\mathrm{q}, \\ \infty & \text{if } p=q=\infty, \end{cases}$$

and

$$\alpha_n = \max_{m,i} \sqrt{\frac{(n+m)!}{(n-m)!}} \,|\, a_{nm}^{(i)} \,|, \qquad i=1,2.$$

PROOF. Fryant [4] has proved that H is an entire function if and only if the function $g(z)=\sum_{n=0}^{\infty}\alpha_n(1+n^{-1/2})^n z^n$ is entire. Applying Theorem 1 of Juneja *et al.* [5] to the function $g(z)$ and after simple manipulations, Lemma 1 follows.

LEMMA 2. *If H is an entire harmonic function having (p,q)-order ρ $(b<\rho<\infty)$, then the generalized (p,q)-type T^* of H is given by*

$$\frac{T^*}{M^*} = \limsup_{n\to\infty} \left[\frac{F(\log^{[p-2]} \eta}{\log^{[q-1]} \alpha_n^{-1/n}} \right]^{\rho-A},$$

where b bas been defined in (1.3), $A=1$ if $q=2$, $A=0$ if $q\neq 2$ and

$$M^* \equiv M^*(p,q) = \begin{cases} (\rho-1)^{\rho-1}/\rho^{\rho} & \text{if } (p,q)=(2,2), \\ 1/ep & \text{if } (p,q)=(2,1), \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. Applying Theorem 3 of Nandan *et al.* [8] to the function $g(z)$, defined in Lemma 1, we get the result of Lemma 2 after simple manipulations.

LEMMA 3. *Associated Legendre's functions $P_n^m(t)$ satisfy*

$$\max_{-1\le t\le 1} |\, P_n^m(t) \,| \le K \sqrt{\frac{(n+m)!}{(n-m)!}},$$

where K is a constant independent of n and m.

PROOF. It is known from Erdelyi [2] that

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dt^{n+m}}[(t^2-1)^n].$$

Thus, $[P_n^m(t)]^2$ is a polynomial of degree *2n*. Using Theorem 2.2.1 due to Sewel [10], we have

$$\max_{-1\le t\le 1}|P_n^m(t)|^2 \le K2n\int_{-1}^{1}(P_n^m(t))^2\,dt,$$

where K is a constant independent of n and m. It is also known from Sansonne ([9], p. 247)

$$\int_{-1}^{1}(P_n^m(t))^2\,dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}.$$

Combining the above inequalities, Lemma 3 is immediate.

LEMMA 4. *Let* $H\in H_R$. *Then, for any* $R_*<R$ *and* $n\ge 1$, *we have*

$$R_*^n\max_{m,i}\left[\left|a_{nm}^{(i)}\right|\left(\frac{(n+m)!}{(n-m)!}\right)^{1/2}\right]\le k_0\,(2n+1)E_{n-1}^{\beta}(H,R),$$

where k_0 *is a constant.*

PROOF. For $H\in H_R$, it is known from Fryant [3] that

$$a_{nm}^{(i)}R_*^n = \frac{2n+1}{2\pi\alpha_m}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi}H(R_*,\theta,\phi)P_n^m(\cos\theta)T_{i,m}(\phi)\sin\theta\,d\theta\,d\phi$$

for every $R_*<R$. Here $\alpha_m=2$ if $m=0$, $\alpha_m=1$ otherwise, $T_{1,m}(\phi)=\cos m\phi$, and $T_{2,m}(\phi)=\sin m\phi$. Thus, for $g\in\pi_{n-1}$

$$a_{nm}^{(i)}R_*^n = \frac{2n+1}{2\pi\alpha_m}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{2\pi}(H(R_*,\theta,\phi)-g(R_*,\theta,\phi))P_n^m(\cos\theta)T_{i,m}(\phi)\sin\theta\,d\theta\,d\phi$$

Using Lemma 3, we get

(2.1) $$\left|a_{nm}^{(i)}\right|R_*^n \le (2n+1)k\pi\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2}\|h-g\|_{\beta,R}.$$

By the definition of $E_n^{\beta}(H,R)$ there exists a $\tilde g\in\pi_{n-1}$ such that

(2.2) $$\|H-\tilde g\|\le 2E_{n-1}^{\beta}(H,R).$$

Taking in particular, $g=\tilde g$ in (2.1), and then using (2.2), we have the result.

Let $w=\psi(z)$ be the univalent function mapping the complement of $\overline{D}_R$ on $|w|>1$ such that $\psi(\infty)=\infty$ and $\psi'(\infty)>0$. Set $D_r=\{z:|\psi(z)|=r,\ r>1\}$. Then,

LEMMA 5. *Let* $H\in H_R$ *be an entire harmonic function of* (p,q)*-order* ρ *and* (p,q)*-type* T^* with respect to $\rho(r)$. Then

$$\limsup_{r\to\infty}\frac{\log^{[p]}\overline{M}(r,H)}{\log^{[q]}r}=\rho,$$

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}\overline{M}(r,H)}{(\log^{[q-1]}r)^{\rho(r)}}=\frac{T^*}{\gamma},$$

where $\overline{M}(r,H)=\max_{z\in D_r}|H|$, $\gamma=R^{-\rho}$ for $q=1$ and $\gamma=1$, otherwise.

The lemma follows on the lines of proof of Lemma 3.1 [11].

LEMMA 6. *Let* $H\in H_R$ *be an entire harmonic function and* $r'>1$. *Then, for all sufficiently large values of n, we have*

$$E_n^{\beta}(H,R)\le K_0\overline{M}(r,H)\left(\frac{r'R}{r}\right)^{n+1},$$

where K_0 *is a constant independent of n and* $r>2r'R$.

PROOF. Let

$$Q_n=\sum_{k=0}^{n}r^k\sum_{m=0}^{k}\left(a_{km}^{(1)}\cos m\phi+a_{km}^{(2)}\sin m\phi\right)P_k^m(\cos\theta).$$

Then $Q_n\in\pi_n$. Now, using the definition of $E_n^{\beta}(H,R)$ and Lemma 3, we get

$$\text{(2.3)}\qquad E_n^{\beta}(H,R)\le|H-Q_n|_{\beta,R}\le$$
$$\le\sum_{k=n+1}^{\infty}R^k\sum_{m=0}^{k}\left|\left(a_{km}^{(1)}\cos m\phi+a_{km}^{(2)}\sin m\phi\right)P_k^m(\cos\theta)\right|\le$$
$$\le K_0\sum_{k=n+1}^{\infty}R^k(2k+1)\max_{m,i}\left|a_{km}^{(i)}\right|\left(\frac{(k+m)!}{(k-m)!}\right)^{1/2}.$$

For an entire harmonic function H ([3], eqn. 2.3),

$$\text{(2.4)}\qquad \max_{m,i}\left|a_{km}^{(i)}\right|\left(\frac{(k+m)!}{(k-m)!}\right)^{1/2}\le 2\sqrt{2k+1}\,\frac{\overline{M}(r,H)}{r^k}.$$

Combining (2.3) and (2.4), we get'

$$E_n^{\beta}(H,R) \le 2K_0\overline{M}(r,H)\sum_{k=n+1}^{\infty}(2k+1)^{3/2}(R/r)^k, \qquad z\in D_r.$$

Thus, for $n > n_0$ and $r > 2r'R$, the above inequality gives

$$E_n^{\beta}(H,R) \le 2K_0\overline{M}(r,H)\sum_{k=n+1}^{\infty}(r'R/r)^k =$$

$$= 2K_0\overline{M}(r,H)\frac{(r'R/r)^{n+1}}{1-(r'R/r)}, \qquad z\in D_r.$$

Hence the lemma.

LEMMA 7. *Let $H\in H_R$ be an entire harmonic function. Then*

$$h(z) = \sum_{n=1}^{\infty}(2n+1)^2 E_{n-1}^{\beta}(H,R)\left(\frac{z}{R_*}\right)^n,$$

is entire. Further, $\rho(H)=\rho(h)$ and for $b<\rho(H)=\rho(h)<\infty$, $T^*(H)=\gamma T^*(h)$, b and γ have been defined in (1.3) *and Lemma 5, respectively.*

PROOF. Since $(2n+1)^{1/n}\to 1$ as $n\to\infty$, it follows from Lemma 6 that $h(z)$ is entire and

$$E_{n-1}^{\beta}(H,R) \le K_0\overline{M}(r+1,H)\left(\frac{r'R}{r+1}\right)^n,$$

we have

$$h(z) = \sum_{n=1}^{\infty}(2n+1)^2 E_{n-1}^{\beta}(H,R)\left(\frac{z}{R_*}\right)^n,$$

which gives

$$(2.5)\quad M(r/Rr',h) \le A(r) + K_0\overline{M}(r+1,H)\sum_{n=0}^{\infty}\left[\frac{r}{R^*(r+1)}\right]^n =$$

$$= A(r) + K_0\frac{R^*(r+1)\overline{M}(r+1,H)}{(r+1)R^*-r}, \qquad r'>1,$$

where $A(r)$ is a polynomial for all sufficiently large values of r.

On the other hand, using (1.2), Lemmas 3 and 4, we get

$$\sum_{n=0}^{n} r^n \sum_{m=0}^{n}\left(a_{nm}^{(1)}\cos m\phi + a_{nm}^{(2)}\sin m\phi\right)P_n^m(\cos\theta) \le$$

$$\leq \left| a_{00}^{(1)} \right| + K \sum_{n=1}^{\infty} (2n+1) r^n \max_{m,i} \left[\left| a_{nm}^{(i)} \right| \left(\frac{(n+m)!}{(n-m)!} \right)^{1/2} \right] \leq$$

$$\leq \left| a_{00}^{(1)} \right| + K K_0 \sum_{n=1}^{\infty} (2n+1)^2 E_{n-1}^{\beta}(H,R) \left(\frac{r}{R_*} \right)^n, \qquad z \in D_r, \quad R_* < R$$

or

$$M(r,H) \leq M\left(\frac{r}{R_*}, \left| a_{00}^{(1)} \right| + K K_0 h(z) \right). \tag{2.6}$$

Now, proof concludes in view of (2.5) and (2.6).

3. Main Results

Theorem 1. *Let* $H \in H_R$, $r > 2r'R$. *Then* H *has an analytic continuation as an entire function, if and only if*

$$\lim_{n \to \infty} [E_n^{\beta}(H,R)]^{1/n} = 0. \tag{3.1}$$

Proof. First suppose that H is entire. Then, it follows from Lemma 6 that

$$\limsup_{n \to \infty} [E_n^{\beta}(H,R)]^{1/n} \leq r'R/r, \qquad r > 2r'R.$$

Thus, for all sufficiently large r, we have

$$\limsup_{n \to \infty} [E_n^{\beta}(H,R)]^{1/n} = 0.$$

Conversely if (3.1) holds, then it follows from (2.6) that the series on the right hand of (1.2) converges uniformly on every compact subset of D_∞ and H is entire.

Theorem 2. *Let* $H \in H_R$, $r > 2r'R$. *Then* H *has analytic continuation as an entire harmonic function of finite* (p,q)*-order* ρ *if and only if*

$$\rho(p,q) = P(L^*(p,q)),$$

where

$$L^*(p,q) = \limsup_{n \to \infty} \frac{\log^{[p-1]} n}{\log^{[q]} \left[E_n^{\beta}(H,R) \right]^{-1/n}}.$$

Proof. By Theorem 1, we have concluded that $H \in H_R$, has an analytic continuation as an entire function if and only if $h(z)$ is an entire function. Moreover, by Lemma 7, H and $h(z)$ have same (p,q)-order. The remaining part of the proof of Theorem 2 is straightforward.

THEOREM 3. *Let $H \in H_R$. Then H has an analytic continuation as an entire harmonic function of (p,q)-order ρ $(b<\rho<\infty)$ and (p,q)-type T^* of H with respect to a proximate order $\rho(r)$ if and only if*

$$\limsup_{n\to\infty}\left[\frac{F\left(\log^{[p-2]} n\right)}{\log^{[q-1]}\left[E_n^{\beta}(H,R)\right]^{-1/n}}\right]^{\rho-A} = \frac{T^*}{M^*\gamma}.$$

PROOF. To prove this theorem, we apply Lemma 2 to the function $h(z)$ and resulting characterization *of* T^* in terms of $E_n^{\beta}(H,R)$ and the relation $T^*=\gamma T^*(h)$, in view of Lemma 7, taking together proves the theorem.

REFERENCES

[1] S.M. Einstein-Matthews, H.S. Kasana, Proximate order and type of entire functions of several complex variables, *Israel J. Math.* (Jerusalem), 92(1-3) (1995), 273-284.

[2] A. Erdelyi *et al.*, *Higher Transcendental Functions*, Vol. I, McGraw-Hill book Co. Inc., New York, 1953.

[3] A.J. Fryant, Growth of entire harmonic functions in R^3, *J. Math. Anal. Appl.* 66(1978), 599-605.

[4] A.J. Fryant, Spherical harmonic expansions, *Pure Appl. Math. Sci.* 22(1985), 25-31.

[5] O.P. Juneja, G.P. Kapoor, S.K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, *J. Reine Angew. Math.* 282(1976), 53-67.

[6] O.P. Juneja, G.P. Kapoor, S.K. Bajpai, On the (p,q)-type and lower (p,q)-type of an entire function, *J. Reine Angew. Math.* 290(1977), 180-190.

[7] H.S. Kasana, A. Sahai, The proximate order of entire Dirichlet series, *Complex Variables: Theory and Application* (New York-USA) 9(1)(1987), 49-62.

[8] K. Nandan, R.P. Doherey, R.S.L. Srivastave, On the generalized type and lower generalized lower type of an entrie function with index-pair (p,q), *Indian J. Pure Appl. Math.* 11(1980), 1424-1433.

[9] G. Sansonne, *Orthogonal Functions,* Interscience, New York, 1959.

[10] W.E. Sewel, *Degree of Approximation by Polynomials in the Complex Domain*, Ann. of Math., Studies no. 9, Princeton University Press, Princeton, 1942.

[11] T. Winiarski, Approximation and interpolation of entire functions, *Ann. Polon. Math.* 23(1970), 259-273.

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