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ON SOME BASIC FINITE DIFFERENCE INEQUALITIES

ABSTRACT: In this paper we establish some basic finite difference inequalities which can be used as tools in the study of qualitative properties of the solutions of finite difference equations and numerical analysis.

KEY WORDS: finite difference inequalities, qualitative properties, finite difference equations, numerical analysis, explicit bound, perturbation, fundamental solution matrix.

1. INTRODUCTION

In the past few years, finite difference inequalities have gained considerable importance in the analysis of various aspects of the theory of finite difference equations and numerical analysis. This led several authors to the discovery of new finite difference inequalities which find important applications in the development of the equalitative theory of finite difference equations, see [1-8] and the references cited therein. The aim of the present paper is to establish some basic finite difference inequalities which provide explicit bounds on unknown functions and offers an effective tool in the theory of finite difference equations and numerical analysis.

2. MAIN RESULTS

In what follows, R denotes the set of real numbers and $R_+ = [0,\infty)$, $N_0 = \{0,1,2,\ldots\}$ are the given subsets of R. We define the operators Δ by $\Delta u(n) = u(n+1) - u(n)$ for any function u defined on N_0. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively.

A useful and important finite difference inequality is established in the following theorem.

THEOREM 1. *Let $u(n)$, $a(n)$ and $b(n)$ be nonneqative functions defined for $n \in N_0$ and*

(1) $$\Delta u(n) \leq a(n)u(n) + b(n).$$

for $n \in N_0$. Then

(2) $$u(n) \leq u(0)\prod_{s=0}^{n-1}[1+a(s)] + \sum_{s=0}^{n-1} b(s) \sum_{\sigma=s+1}^{n-1}[1+a(\sigma)],$$

for $n \in N_0$.

PROOF. From (1) we observe that

(3) $$u(n+1)-[1+a(n)]u(n) \leq b(n).$$

Multiplying both sides of (3) by $\prod_{\sigma=0}^{n}[1+a(\sigma)]^{-1}$, we have

(4) $$u(n+1)\prod_{\sigma=0}^{n-1}[1+a(\sigma)]^{-1} - u(n)\prod_{\sigma=0}^{n-1}[1+a(\sigma)]^{-1} \leq b(n)\prod_{\sigma=0}^{n-1}[1+a(\sigma)]^{-1}.$$

By setting $n = s$ in (4) and taking the sum over s from 0 to $n-1$ we get

(5) $$u(n)\prod_{\sigma=0}^{n-1}[1+a(\sigma)]^{-1} - u(0) \leq \sum_{s=0}^{n-1} b(s)\prod_{\sigma=0}^{s}[1+a(\sigma)]^{-1}.$$

The required inequality in (2) follows from (5).

REMARK. If we take $u(0)=0$ in Theorem 1, then the bound obtained in (2) reduces to

$$u(n) \leq \sum_{s=0}^{n-1} b(s)\prod_{\sigma=s+1}^{n-1}[1+a(\sigma)],$$

for $n \in N_0$.

Another interesting and useful finite difference inequality is given in the following theorem.

THEOREM 2. *Let* $u(n)$, $a(n)$ *and* $b(n)$ *be nonnegative functions defined for* $n \in N_0$ *and* $\Delta a(n) \geq 0$ for $n \in N_0$. *If*

(6) $$u(n) \leq a(n) + \sum_{s=0}^{n-1} b(s)u(s),$$

for $n \in N_0$, *then*

(7) $$u(n) \leq a(0)\sum_{s=0}^{n-1}[1+b(s)] + \sum_{s=0}^{n-1}\Delta a(s)\prod_{\sigma=s+1}^{n-1}[1+b(\sigma)],$$

for $n \in N_0$.

PROOF. Define a function $z(n)$ by the right hand side of (6). Then $z(0)=a(0)$, $u(n)\le z(n)$ and

$$\Delta z(n) = \Delta a(n)+ b(n)u(n)\le b(n)z(n) +\Delta a(n) \tag{8}$$

Now by applying Theorem 1 to (8) we get

$$z(n) \le a(0)\sum_{s=0}^{n-1} [1+b(s)] + \sum_{s=0}^{n-1} \Delta a(s) \prod_{\sigma=s+1}^{n-1} [1+b(\sigma)]. \tag{9}$$

Using (9) in $u(n)\le z(n)$ we get the required inequality in (7).

The inequalities established in the following theorems can be used in certain situations.

THEOREM 3. *Let $u(n)$, $a(n)$, $b(n)$, $c(n)$, $p(n)$ be nonnegative functions defined for $n\in N_0$ and $\Delta c(n)\ge 0$ for $n\in N_0$. If*

$$u(n) \le a(n) + b(n)\left(c(n) + \sum_{s=0}^{n-1} p(s)u(s) \right), \tag{10}$$

for $n\in N_0$, then

$$u(n) \le a(n) + b(n)\Bigg[c(0)\prod_{s=0}^{n-1} [1+b(s)p(s)] + \\ + \sum_{s=0}^{n-1} [\Delta c(s)+ a(s)p(s)] \prod_{\sigma=s+1}^{n-1} [1+b(s)p(s)] \Bigg], \tag{11}$$

for $n\in N_0$.

PROOF. Define a function $z(n)$ by

$$z(n) = c(n) + \sum_{s=0}^{n-1} p(s)u(s). \tag{12}$$

Then $z(0)=c(0)$ and

$$z(n) \le a(n) + b(n)z(n). \tag{13}$$

From (12) and (13) we have

$$\Delta z(n) = \Delta c(n) + p(n)u(n) \le \\ \le b(n)p(n)z(n) + [\Delta c(n) + a(n)p(n)]. \tag{14}$$

Now by applying Theorem 1 to (14) we get

(15) $$z(n) \le c(0)\prod_{s=0}^{n-1}[1+b(s)p(s)] + \sum_{s=0}^{n-1}[\Delta c(s) + a(s)p(s)] \times \times \prod_{\sigma=s+1}^{n-1}[1+b(s)p(s)].$$

Using (15) in (13) we get the required inequality in (11).

REMARK 2. We note that in the special case when $c(n)=0$, the inequality given in Theorem 3 reduces to the inequality given by Pachpatte in [5, Theorem 1].

THEOREM 4. *Let* $u(n)$, $a(n)$, $b(n)$, $c(n)$, $\Delta c(n)$ *be as in Theorem 3. Let* $L : N_0 \times R_+ \to R_+$ *be a function such that*

(16) $$0 \le L(n,x) - L(n,y) \le M(n,y)(x-y),$$

for $n \in N_0$, $x \ge y \ge 0$, *where* $M(n,y)$ *is a real-valued nonneqative function defined for* $n \in N_0$, $y \in R_+$. *If*

(17) $$u(n) \le a(n) + b(n)\left(c(n) + \sum_{s=0}^{n-1} L(s,u(s)) \right),$$

for $n \in N_0$, *then*

(18) $$u(n) \le a(n) + b(n)\left[c(0)\prod_{s=0}^{n-1}[1+M(s,a(s))b(s)] + \sum_{s=0}^{n-1}[\Delta c(s) + L(s,a(s))] \prod_{\sigma=s+1}^{n-1}[1+M(\sigma,a(\sigma))b(\sigma))\right],$$

for $n \in N_0$.

PROOF. Define a function $z(n)$ by

(19) $$z(n) = c(n) + \sum_{s=0}^{n-1} L(s,u(s)).$$

Then $z(0)=c(0)$ and

(20) $$u(n) \le a(n) + b(n)z(n).$$

From (19), (20) and (16) we have

(21) $$\begin{aligned} \Delta z(n) &= \Delta c(n) + L(n,u(n)) \le \\ &\le \Delta c(n) + L(n,a(n)+b(n)z(n)) - L(n,a(n)) + L(n,a(n)) \le \\ &\le M(n,a(n))b(n)z(n)) + [\Delta c(n) + L(n,a(n))]. \end{aligned}$$

Now applying Theorem 1 to (21) we get

$$z(n) \leq c(0) \prod_{s=0}^{n-1} [1+M(s,a(s))b(s)] + \sum_{s=0}^{n-1} [\Delta c(s) + L(s,a(s))] \times \prod_{\sigma=s+1}^{n-1} [1+M(\sigma,a(\sigma))b(\sigma)]. \tag{22}$$

Using (22) in (20) we get the desired inequality in (18).

THEOREM 5. *Let* $u(n)$, $a(n)$, $b(n)$, $c(n)$, $\Delta c(n)$ *be as in Theorem 3. Let* $L : N_0 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$0 \leq L(n,x) - L(n,y) \leq M(n,y)\phi^{-1}(x-y), \tag{23}$$

for $n \in N_0$, $x \geq y \geq 0$, *where* $M(n,y)$ *is defined as in Theorem 4,* $\phi : R_+ \to R_+$ *is a continuous and strictly increasing function with* $\phi(0)=0$, ϕ^{-1} *is the inverse function of* ϕ, *and*

$$\phi^{-1}(xy) \leq \phi^{-1}(x)\phi^{-1}(y), \tag{24}$$

for $x, y \in R_+$. *If*

$$u(n) \leq a(n) + b(n)\phi\left(c(n) + \sum_{s=0}^{n-1} L(s,u(s)) \right), \tag{25}$$

for $n \in N_0$, *then*

$$u(n) \leq a(n) + b(n)\phi\left(c(0) \prod_{s=0}^{n-1} [1+M(s,a(s))\phi^{-1}(b(s))] + \sum_{s=0}^{n-1} [\Delta c(s) + L(s,a(s))] \prod_{\sigma=s+1}^{n-1} [1+M(\sigma,a(\sigma))\phi^{-1}(b(\sigma))] \right), \tag{26}$$

for $n \in N_0$.

PROOF. Define a function $z(n)$ by (19). Then $z(0)=c(0)$ and

$$u(n) \leq a(n) + b(n)\phi(z(n)). \tag{27}$$

From (19), (27), (23) and (24) we have

$$\begin{aligned}\Delta z(n) &= \Delta c(n) + L(n,u(n)) \leq \\ &\leq \Delta c(n) + L(n,a(n) + b(n)\phi(z(n))) - L(n,a(n)) + L(n,a(n)) \leq \\ &\leq M(n,a(n))\phi^{-1}(b(n)\phi(z(n))) + [\Delta c(n) + L(n,a(n))] \leq \\ &\leq M(n,a(n))\phi^{-1}(b(n))z(n) + [\Delta c(n) + L(n,a(n))].\end{aligned}$$

Now an application of Theorem 1 to (28) we get

$$z(n) \le c(0)\prod_{s=0}^{n-1}\left[1 + M(s,a(s))\phi^{-1}(b(s))\right] +$$

$$+\sum_{s=0}^{n-1}[\Delta c(s) + L(s,a(s))]\prod_{\sigma=s+1}^{n-1}\left[1 + M(\sigma,a(\sigma))\phi^{-1}(b(\sigma))\right].$$

Using (29) in (27) we get (26).

3. An Application

In this section, we present an anpplication of Theorem 3 to obtain an explicit bound on solution of the following system of difference equation

$$(P) \qquad x(n+1) = A(n)x(n) + f(n,x(n)) + r(n), \qquad x(n_0) = x_0,$$

considered as a perturbation of the linear system

$$(L) \qquad y(n+1) = A(n)y(n), \qquad y(n_0) = x_0,$$

where n_0, $n \in N_0$, x, y, f, r are the elements of R^n, the n dimensional vector space, $A(n)$ is an $n \times n$ matrix with $\det A(n) \ne 0$, the functions r and f are defined on N_0 and $N_0 \times R^n$ respectively and x_0 is a given vector in R^n. The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix norm. We denote by $x(n)$ and $y(n)$ the solutions of (P) and (L) respectively and $Y(n)$ denote the fundamental solution matrix of the system (L) such that $Y(n_0) = I$, the identity matrix.

The following theorem deals with the bound on the solution of (P) under some suitable conditions on the function f involved in (P) and the fundamental solution matrix of (L).

Theorem 6. *Assume that the fundamental solution matrix $Y(n)$ of (L) satisfies*

$$(30) \qquad |Y(n)Y^{-1}(s)| \le b(n),$$

for $n_0 \le s \le n$, $n_0, s, n \in N_0$ and the function f in (P) satisfies

$$(31) \qquad |f(n,x)| \le p(n)|x|,$$

for $n \in N_0$, $x \in R^n$, where b and p are defined as in Theorem 3. Then

$$(32) \qquad |x(n)| \le b(n)\left[|x_0| + \sum_{s=n_0}^{n-1}[|r(s)| + |x_0|b(s)p(s)]\prod_{\sigma=s+1}^{n-1}[1 + b(s)p(s)]\right],$$

for $n \in N_0$.

PROOF. By using the variation of constants formula any solution $x(n)$ of (P) is represented by

$$x(n) = Y(n)Y^{-1}(n_0)x_0 + \sum_{s=n_0}^{n-1} Y(n)Y^{-1}(s+1)[f(s,x(s))) + r(s)]. \tag{33}$$

Using (30), (31) in (33) we obtain

$$|x(n)| \leq b(n)|x_0| + b(n)\left(\sum_{s=n_0}^{n-1} |r(s)| + \sum_{s=n_0}^{n-1} p(s)|r(s)|\right). \tag{34}$$

Now a suitable application of Theorem 3 to (34) yields the required estimation in (32).

In concluding we note that the inequalities established here can be used to study various properties of the solutions of certain classes of finite difference and sum-difference equations. The multidimensional versions of these inequalities and their applications will be reported elsewhere.

REFERENCES

[1] D. Bainov, P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[2] P.R. Beesak, *Gronwall Inequalities*, Carleton Mathematical Lecture Notes, No. 11, 1975.

[3] C.V. Coffman, Asymptotic behavior of solutions of ordinary difference equations, *Trans. Amer. Math. Soc.*, 110(1964), 22-51.

[4] F. Ma, *Stablity theory of Stochastic difference systems, in „Probabilistic Analysis and related topics" A.T. Bharucha-Reid Ed.,* vol. 3, 127-160, Academic Press, New York, 1983.

[5] B.G. Pachpatte, On the discrete generalizations of Gronwall's inequality, *J. Indian Math. Soc.*, 37(1973), 147-156.

[6] B.G. Pachpatte, Some new finite difference inequalities, *Computers Math. Applic.*, 28(1994), 227-241.

[7] B.G. Pachpatte, On some new inequalities related to a certain inequality arising in the theory of differential equations, *J. Math. Anal. Appl.* 251(2000), 736-751.

[8] G. Papaschinopoulos, On the summable manifold for discrete systems, *Math. Japonica* 33(1998), 451-468.

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