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KAROLINA TOMCZAK

SOME REMARKS OF BOREL TYPE MEANS OF TRIGONOMETRIC FOURIER SERIES

ABSTRACT: In this note we consider certain means $B_i(f)$ of Borel type of trigonometric Fourier series of function f belonging to the Hölder space $H_{2\pi}^{\omega}$.

We give theorem on the degree of approximation of $f \in H_{2\pi}^{\omega}$ by $B_i(f)$.

KEY WORDS: trigonometric Fourier series, Borel mean, Hölder space, degree of approximation.

1. PRELIMINARIES

1.1. Let $C_{2\pi}$ be the space of 2π-periodic real-valued functions f, continuous on $J=[-\pi;\pi]$ with the norm

$$(1) \qquad \|f\|_c := \max_{x\in J} |f(x)|.$$

For the function $f \in C_{2\pi}$ we define the modulus of smoothness [3]

$$(2) \qquad \omega_2(t;f) := \sup_{|h|\le t} \|\Delta_h^2 f(\cdot)\|_C, \qquad t\in[0,\pi],$$

where

$$(3) \qquad \Delta_h^2 f(x) := f(x+h)+f(x-h)-2f(x).$$

1.2. Similarly as in [1] and [2] we denote by Ω the set of all functions of the type of modulus smoothness of order 2, i.e. Ω is the set of all functions m having the following properties:

(i) ω is defined and continuous on $[0,+\infty)$,

(ii) ω is monotonically increasing and $\omega(0)=0$,

(iii) $\omega(t)t^{-2}$ is monotonically decreasing for $t>0$.

As in [1] and [2], for a given $\omega\in\Omega$, we denote by $H_{2\pi}^{\omega}$ the space of all functions $f\in C_{2\pi}$ for which the quantity

$$(4) \qquad \|f\|_{\omega}^{*} := \sup_{h>0} \frac{\|\Delta_h^2 f(\cdot)\|_C}{\omega(h)}$$

is finite and the norm is defined by the formula

(5) $$\| f \|_{H^{\omega}} := \| f \|_C + \| f \|_{\omega}^{*}.$$

Denote by $\widetilde{H}_{2\pi}^{\omega}$ all functions $f \in H_{2\pi}^{\omega}$ for which

(6) $$\lim_{h\to 0^+} \frac{\| \Delta_h^2 f(\cdot) \|_C}{\omega(h)} = 0$$

and the norm is defined by (5). $H_{2\pi}^{\omega}$ and $\widetilde{H}_{2\pi}^{\omega}$ are called generalized Hölder spaces.

If $\omega(t) = t^{\alpha}$, $t \in [0,\pi]$, with a fixed $0 < \alpha \le 2$, then $H_{2\pi}^{\omega}$ and $\widetilde{H}_{2\pi}^{\omega}$ are classical Hölder spaces.

We observe that if $\mu, \omega \in \Omega$ and the function

(7) $$\lambda(t) := \frac{\omega(t)}{\mu(t)}, \qquad t \in (0,\pi],$$

is monotonically increasing , then

(8) $$H_{2\pi}^{\omega} \subset H_{2\pi}^{\mu}, \qquad \widetilde{H}_{2\pi}^{\omega} \subset \widetilde{H}_{2\pi}^{\mu}.$$

If $f \in H_{2\pi}^{\omega}$, then for modulus $\omega(\cdot; f)$ defined by (2) we have

(9) $$\omega_2(t; f) \le \omega(t) \| f \|_w^{*} \quad \text{for} \quad t \in (0,\pi].$$

1.3. Let $f \in C_{2\pi}$ and let

(10) $$f(x) \sim \frac{1}{2} a_0(f) + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)$$

be its Fourier series. Let $S_n(\cdot; f)$, $n \in N := \{0,1,2,\ldots\}$ be the n-th partial sum of the series (10).

It is easily proved that for every $f \in C_{2\pi}$ there exist means $B_i(\,\cdot\,;\,\cdot\,;f)$, $i = 1,2$, of the Borel type of the Fourier series (9):

(11) $$B_i(r,x;f) := \sum_{k=0}^{\infty} p_{i,k}(r) S_k(x;f), \qquad x \in R, \quad r \in R_+,$$

where $R := (-\infty, +\infty)$, $R_+ := (0, +\infty)$

(12) $$p_{1,k}(r) := \frac{1}{\cosh r} \frac{r^{2k}}{(2k)!}, \quad p_{2,k}(r) := \frac{1}{\sinh r} \frac{r^{2k+1}}{(2k+1)!}, \qquad r \in R_+, \quad k \in N,$$

and $\cosh r$, $\sinh r$ are elementary hyperbolic functions.

2. (B_i) - MEANS IN THE SPACE $C_{2\pi}$

Some approximation properties of $B_i(f)-$ means of Fourier series (10) of $f \in C_{2\pi}$ were given in [4]. In the paper [4] was derived the following integral formula for $B_i(f)$ detined by (11) and (12):

$$B_i(r,x;f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_i(r;t)\,dt\,, \quad x \in R\,, \quad r \in R_+\,, \quad i=1,2, \tag{13}$$

where

$$K_i(r,t) := \sum_{n=0}^{\infty} p_{i,n}(r)\,D_n(t)\,, \tag{14}$$

$$D_0 = \frac{1}{2}, \qquad D_n(t) = \frac{1}{2}+\sum_{j=1}^{n} \cos jt \quad \text{for} \quad n \geq 1,$$

and consequently

$$K_1(r,t) = \frac{1}{2\cosh r \sinh\frac{t}{2}}[\cos\tfrac{t}{2}\sin(r\sin\tfrac{t}{2})\sinh(r\cos\tfrac{t}{2}) + \\ + \sin\tfrac{t}{2}\cos(r\sin\tfrac{t}{2})\cosh(r\cos\tfrac{t}{2})]\,,$$

$$K_2(r,t) = \frac{\sin(r\sin\frac{t}{2})\cosh(r\cos\frac{t}{2})}{2\sinh r \sin\frac{t}{2}}$$

for $r \in R_+$ and $|\,t\,|\in (0,\pi]$. From this it follows that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_i(r;t)\,dt = 1 \qquad \text{for} \quad r \in R_+\,, \quad i=1,2, \tag{15}$$

and by (3), (13) – (15) we get

$$B_i(r,x;f) - f(x) = \frac{1}{\pi}\int_{0}^{\pi} [\Delta_t^2 f(x)]\,K_i(r,t)\,dt \tag{16}$$

for every $f \in C_{2\pi}$, $x \in R$, $r \in R_+$ and $i=1,2$.

In [4] were proved the following lemma and theorem

LEMMA 1. *Let* $K_i(\cdot,\cdot)$, $i=1,2$, *be function defined by* (14) *or* (15). *Then there exist a positive constant* $M_1 = M_1(i)$ *depending only on i and such that*

$$\int_{-\pi}^{\pi} |K_i(r,t)|dt = 2\int_{0}^{\pi} |K_i(r,t)|dt \le M_1(1+\ln r)$$

for $r>1$ *and* $i=1,2.$

THEOREM 1. *Let* $f \in C_{2\pi}$ *and* $i=1,2.$ *Then there exist positive constants* $M_k = M_k(i),\ k=2,3,$ *depending only on i such that*

(17) $$\|B_i(r,\cdot;f)\|_C \le M_2 \|f\|_C (1+\ln r)$$

and

(18) $$\|B_i(r,\cdot;f)-f(\cdot)\|_C \le M_3\omega_2(r^{-\frac{1}{2}};f)(1+\ln r)$$

for $r>1$ *and* $i=1,2,$ *where* $\omega_2(f)$ *is defined by* (2).

3. (B_i) - MEANS IN HÖLDER SPACES $H_{2\pi}^{\omega}$ AND $\widetilde{H}_{2\pi}^{\omega}$

3.1. First we shall prove the main lemma.

LEMMA 2. *Let* $\omega \in \Omega$ *be a fixed function. Then there exists a positive constant* M_4 *such that*

(19) $$\|B_i(r,\cdot;f)\|_{\omega}^{*} \le M_4 \|f\|_{\omega}^{*} (1+\ln r)$$

for every $f \in H_{2\pi}^{\omega},\ r>1$ *and* $i=1,2.$ *The formula* (13) *and the inequality* (19) *show that if* $f \in H_{2\pi}^{\omega}$, *then for every fixed* $r>1$ *and* $i=1,2$ *the function* $B_i(r,\cdot;f)$ *belongs also to* $H_{2\pi}^{\omega}$.

PROOF. By (3) and (13) we have

(20) $$\Delta_h^2 B_i(r,x;f) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [\Delta_h^2 f(x+t)] K_i(r,t)\,dt$$

for every $f \in H_{2\pi}^{\omega},\ h,r \in R_+,\ x \in R$ and $i=1,2.$ Applying (2) and Lemma 1 we get from (20)

$$\|\Delta_h^2 B_i(r,\cdot;f)\|_C \le \|\Delta_h^2 f\|_C \int_{-\pi}^{\pi} |K_i(r,t)|dt \le M_1(1+\ln r)\|\Delta_h^2 f\|_C$$

for $r>1,\ h>0$ and $i=1,2.$ Hence

$$\frac{\| \Delta_h^2 B_i(r,\cdot;f)\|_C}{\omega(h)} \le M_1(1+\ln r)\frac{\| \Delta_h^2 f \|_C}{\omega(h)}$$

for $r>1$, $h>0$ and $i=1,2$, which by definition (4) immediately implies the desired inequality (19).

Applying (17), Lemma 2 and (4) we derive the following

COROLLARY 1. *Let $\omega \in \Omega$ be a fixed function and let $f \in H_{2\pi}^{\omega}$. Then there exists a positive constant M_5 such that*

$$\| B_i(r,\cdot;f)\|_{H^\omega} \le M_5(1+\ln r)\| f \|_{H^\omega} \quad \textit{for} \quad r>1, \ i=1,2.$$

LEMMA 3. *Let $\omega \in \Omega$ be a fixed function. Then for every $f \in \widetilde{H}_{2\pi}^{\omega}$ and $i=1,2$ and $r>1$ we have $B_i(r,\cdot;f) \in \widetilde{H}_{2\pi}^{\omega}$.*

PROOF. Assume that $f \in \widetilde{H}_{2\pi}^{\omega}$. Then for *f* is satisfied the condition (6). We shall prove that the condition (6) is satisfied also for B_i, $i=1,2$.

By (20) and (16) we have

$$\Delta_h^2 B_i(r,x;f) = B_i(r,x;\Delta_h^2 f(\cdot)),$$

$h>0$, $i=1,2$ and $x \in R$, $r>1$. Hence by (17) we get

$$\| \Delta_h^2 B_i(r,\cdot;f)\|_C \le M_2 \| \Delta_h^2 f(\cdot)\|_C \,(1+\ln r),$$

which implies

$$0 \le \frac{\| \Delta_h^2 B_i(r,\cdot;f)\|_C}{\omega(h)} \le M_2(1+\ln r)\frac{\| \Delta_h^2 f(\cdot)\|_C}{\omega(h)}$$

for $h>0$, $r>1$ and $i=1,2$.

From this inequality and (6) we deduce that

$$\lim_{h\to 0_+} \frac{\| \Delta_h^2 B_i(r,\cdot;f)\|_C}{\omega(h)} = 0$$

for $i=1,2$ and every fixed $r>1$. This proves that $B_i(r,\cdot;f) \in \widetilde{H}_{2\pi}^{\omega}$.

3.2. Now we shall prove the main theorems.

THEOREM 2. *Suppose that $\mu, \omega \in \Omega$ are fixed functions and the function*

$\lambda(\cdot)$, defmed by (7), is increasing on R_+. Then there exists a positive constant $M_6 = M_6(\mu)$ such that for every $f \in H_{2\pi}^{\omega}$ and $r>1$ and $i=1,2$ we have

(21) $$\| B_i(r,\cdot;f)-f(\cdot)\|_{H^\mu} \le M_6 \| f \|_\omega^* (1+\ln r)\lambda(r^{-\frac{1}{2}}).$$

PROOF. By our assumptions and by (5) and (7) we have

(22) $$\| B_i(r,\cdot;f)-f(\cdot)\|_{H^\mu} = \| B_i(r,\cdot;f)-f(\cdot)\|_C + \| B_i(r,\cdot;f)-f(\cdot)\|_\mu^*$$

for $r>1$ and $i=1,2$.

Applying (5), (7) – (9) and (18) we deduce that

(23) $$\| B_i(r,\cdot;f)-f(\cdot)\|_C \le M_3\,\omega(r^{-\frac{1}{2}})(1+\ln r)\| f \|_\omega^* \le$$
$$\le M_3\,\mu(1)\lambda(r^{-\frac{1}{2}})(1+\ln r)\| f \|_\omega^*$$

for $r>1$ and $i=1,2$. Moreover by (4) we can write for $r>1$ and $i=1,2$:

$$\| B_i(r,\cdot;f)-f(\cdot)\|_\mu^* = \sup_{0<h\le\frac{1}{\sqrt{r}}} \frac{\| \Delta_h^2 [B_i(r,\cdot;f)-f(\cdot)]\|_C}{\mu(h)} +$$
$$+ \sup_{h>\frac{1}{\sqrt{r}}} \frac{\| \Delta_h^2 [B_i(r,\cdot;f)-f(\cdot)]\|_C}{\mu(h)} := Z_1 + Z_2 .$$

From (3) and (1) we get $\| \Delta_h^2 g \|_C \le 4 \| g \|_C$ for $g \in C_{2\pi}.g$ Hence, applying (18) and (9) and (7) we can write

$$Z_2 \le 4 \sup_{h>\frac{1}{\sqrt{r}}} \frac{\| B_i(r,\cdot;f)-f(\cdot)\|_C}{\mu(h)} \le 4M_3 \frac{\omega_3(r^{-\frac{1}{2}};f)}{\mu(r^{-\frac{1}{2}})}(1+\ln r) \le$$
$$\le 4M_3 \| f \|_\omega^* \lambda(r^{-\frac{1}{2}})(1+\ln r)$$

for $r>1$ and $i=1,2$. From (3) and (20) and (13) we deduce that

$$\Delta_h^2[B_i(r,x;f)-f(x)] = \Delta_h^2 B_i(r,x;f) - \Delta_h^2 f(x) = B_i(r,x;\Delta_h^2 f) - \Delta_h^2 f(x)$$

for $x \in R$, $r,h \in R_+$ and $i=1,2$. This equality and (17) imply that

$$\| \Delta_h^2[B_i(r,\cdot;f)-f(\cdot)]\|_C \le \| B_i(r,\cdot;\Delta_h^2 f)\|_C + \| \Delta_h^2 f \|_C \le$$
$$\le [M_2(1+\ln r)+1]\| \Delta_h^2 f \|_C$$

for $r>0$, $h>0$ and $i=1,2$. From this and by (4) and (6) it follows that

$$Z_1 \le M_7(1+\ln r) \sup_{0<h\le\frac{1}{\sqrt{r}}} \frac{\| \Delta_h^2 f \|_C}{\mu(h)} \le M_7 \| f \|_\omega^* (1+\ln r)\lambda(r^{-\frac{1}{2}})$$

for $r>1$ and $i=1,2$, where $M_7 = \text{const.}>0$. Consequently we have

$$\| B_i(r,\cdot;f) - f(\cdot) \|_\mu^* \le M_8 \| f \|_\omega^* (1+\ln r)\lambda(r^{-\frac{1}{2}}) \tag{24}$$

for $r>0$ and $i=1,2$, where $M_8 = 4M_3 + M_7$. Now from (22)–(24) follows (21). Thus the proof is completed.

Analogously we can prove the following

THEOREM 3. *Suppose that ω, μ and λ are functions as in Theorem 2. Then for every $f \in \tilde{H}_{2\pi}^\omega$ and $i=1,2$ we have*

$$\lim_{r\to\infty} \frac{\| B_i(r,\cdot;f) - f \|_{H^\mu}}{(1+\ln r)\lambda(r^{-\frac{1}{2}})} = 0\,.$$

From Theorem 2 we can derive the following

COROLLARY 2. *Suppose that ω, μ and λ are functions as in Theorem 2 and moreover exist $M_9 = \text{const.}>0$ and $0<\delta<2$ such that $\lambda(t)\le M_9 t^\delta$ for $t>0$. Then*

$$\| B_i(r,\cdot;f) - f(\cdot) \|_{H^\mu} \le M_{10} \| f \|_\omega^* r^{-\frac{\delta}{2}}(1+\ln r)$$

for all $r>1$ and $i=1,2$, where $M_{10} = M_{10}(\mu(1)) = \text{const.}>0$.

Moreover, if $\omega(t)=t^\alpha$, $\mu(t)=t^\beta$ for all $t>0$ and for fixed $0<\beta<\alpha\le 2$ and if $f \in H_{2\pi}^\omega$, then

$$\| B_i(r,\cdot;f) - f(\cdot)] \|_{H^\mu} \le M_{11} \| f \|_\omega^* r^{\frac{(\beta-\alpha)}{2}}(1+\ln r)$$

for all $r>1$ and $i=1,2$, where $M_{11} = M_{11}(\mu(1)) = \text{const.}>0$.

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