FASCICULI MATHEMATICI

Nr 34 2004

ZBIGNIEW WALCZAK

CERTAIN MODIFICATION OF SZASZ-MIRAKYAN OPERATORS

ABSTRACT: We consider certain modified Szasz-Mirakyan operators $A_n(f;r)$ in space C_0 of uniformly continuous functions. We study approximation properties of these operators.

KEY WORDS: Szasz-Mirakyan operator, degree of approximation, Voronovskaya type theorem.

1. INTRODUCTION

1.1. In the paper [1] were examined approximation properties of Szasz–Mirakyan operators

(1) $$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \quad x\in R_0 = [0,+\infty), \quad n\in N := \{1,2...\},$$

in polynomial weighted spaces C_p, $p\in N_0 := \{0,1,2,...\}$.

If $p=0$, then C_0 is the set of all real–valued functions f uniformly continuous and bounded on R_0 and the norm in C_0 is defined by the formula

(2) $$\| f \| \equiv \| f(\cdot) \| := \sup_{x\in R_0} | f(x) | .$$

In [1] were proved theorems on the degree of approximation of $f \in C_p$ by operators S_n defined by (1). From these theorems was deduced that

(3) $$\lim_{n\to\infty} S_n(f;x) = f(x),$$

for every $f\in C_p$, $p\in N_0$, and $x\in R_0$. Moreover, the convergence (3) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

1.2. In this paper we shall modify the formula (1) and we shall study certain approximation properties of introduced operators.

Let C_0 be the space given in above and let $C_0^1 := \{f \in C_0 : f' \in C_0\}$, where f' is the first derivative of f.

For $f \in C_0$ we define the modulus of continuity $\omega_1(f;\cdot)$ as usual ([2]) by formula

(4) $$\omega_1(f;t) \equiv \omega_1(f;C_0;t) := \sup_{0\le h\le t} \| \Delta_h f(\cdot) \|, \quad t \in R_0,$$

where $\Delta_h f(x) = f(x+h) - f(x)$, for $h, x \in R_0$. From the above it follows that

(5) $$\lim_{t\to 0+} \omega_1(f;t) = 0,$$

for every $f \in C_0$. Moreover, if $f \in C_0^1$ then there exists $M_1 = const. > 0$ such that

(6) $$\omega_1(f;t) \le M_1 \cdot t \qquad \text{for} \quad t \in R_0.$$

1.3. We introduce the operators A_n by the following

Definition 1. *Let* $R_2 := [2,+\infty)$ *and let* $r \in R_2$ *be a fixed number. For function* $f \in C_0$ *we define the operators*

(7) $$A_n(f;r;x) := e^{-(nx+1)^r} \sum_{k=0}^{\infty} \frac{(nx+1)^{rk}}{k!} f\left(\frac{k}{n(nx+1)^{r-1}} \right), \quad x \in R_0, \ n \in N.$$

It is obvious that $A_n(f;r)$ is well defined for every $f \in C_0$ and $n \in N$. Moreover from (7) we easily derive the following formulas

(8) $$A_n(1;r;x) = 1,$$

$$A_n(t;r;x) = x + \frac{1}{n},$$

$$A_n(t^2;r;x) = \left(x + \frac{1}{n} \right)^2 \left[1 + \frac{1}{(nx+1)^r} \right],$$

for every fixed $r \in R_2$ and for all $n \in N$ and $x \in R_0$.

2. Main results

2.1. From formulas (7), (8) and $A_n(t^k;r;x)$, $k = 1,2$, given in the above we obtain

LEMMA 1. *Let* $r \in R_2$ *be a fixed number. Then for all* $x \in R_0$ *and* $n \in N$ *we have*

$$A_n(t-x;r;x) = \frac{1}{n},$$

$$A_n((t-x)^2;r;x) = \frac{1}{n^2}\left[1+\frac{1}{(nx+1)^{r-2}}\right],$$

and

$$\| A_n((t-\cdot)^2;r;\cdot)\| \equiv \sup_{x\in R_0} A_n((t-x)^2;r;x) \le \frac{2}{n^2} \qquad for \quad n\in N.$$

Moreover, by the Hölder inequality and by (7) *and* (8) *we have*

$$\| A_n(|t-\cdot|;r;\cdot)\|^2 \le \| A_n((t-\cdot)^2;r;\cdot)\| \quad for\ n\in N.$$

Now we shall prove the main lemma.

LEMMA 2. *Let* $r \in R_2$ *be a fixed number. Then for every* $f \in C_0$ *and* $n \in N$ *we have*

$$\|A_n(f;r;\cdot)\| \le \| f \|. \tag{9}$$

The formula (7) *and the inequality* (9) *show that* $A_n(f;r;\cdot)$, $n\in N$, $r\in R_2$, *is a positive linear operator from the space* C_0 *into* C_0.

PROOF. From (7) and (2) we deduce that

$$\|A_n(f;r;\cdot)\| \le \| f \| \| A_n(1;r;\cdot)\|$$

for $f \in C_0$, $n \in N$ and $r \in R_2$. Now applying (8), we obtain (9).

2.2. In this section we shall give three theorems on the degree of approximation of $f \in C_0$ by A_n.

THEOREM 1. *If* $f \in C_0^1$ *and* $r \in R_2$ *is a fixed number, then*

$$\| A_n(f;r;\cdot) - f(\cdot)\| \le \frac{\sqrt{2}}{n}\| f' \|, \qquad n\in N. \tag{10}$$

PROOF. Let $x \in R_0$ be a fixed point. Then for $f \in C_0^1$ we have

$$f(t)-f(x) = \int_x^t f'(u)du, \qquad t\in R_0.$$

From this and by (7) and (8) we get

$$A_n(f(t);r;x)-f(x) = A_n\left(\int_x^t f'(u)du;r;x\right), \qquad n\in N.$$

But by (2) we have

$$\left|\int_x^t f'(u)du\right| \le \|f'\|\,|t-x|, \qquad t,x\in R_0,$$

which implies

(11) $$\|A_n(f(t);r;x)-f(x)\| = \|f'\|\,A_n(|t-x|;r;)$$

for $n\in N$. Applying Lemma 1, we get

$$A_n(|t-x|;r;x) \le \frac{\sqrt{2}}{n}, \qquad n\in N.$$

From this and by (11) we immediately obtain (10).

THEOREM 2. *Let* $r\in R_2$ *be a fixed number and let* $f\in C_0$. *Then*

(12) $$\|A_n(f;r;\cdot)-f(\cdot)\| \le 3\omega_1\left(f;\frac{\sqrt{2}}{n}\right) \quad \text{for all } n\in N.$$

PROOF. In this proof we shall use the Stieklov function

(13) $$f_h(x) := \frac{1}{h}\int_0^h f(x+t)dt, \quad x\in R_0,\ h>0,$$

of function $f\in C_0$. From (13) we get

$$f_h(x)-f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)dt,$$

$$f_h'(x) = \frac{1}{h}\Delta_h f(x), \quad x\in R_0,\ h>0.$$

Consequently

(14) $$\|f_h-f\| \le \omega_1(f;h),$$

(15) $$\| f_h' \| \le h^{-1}\,\omega_1(f;h),$$

for $h>0$ and we see that $f_h \in C_0^1$ if $f \in C_0$. Hence, for $x\in R_0$ and $n\in N$, we can write

(16) $$A_n(f;r;x) - f(x) \le A_n(f - f_h;r;x) + \\ + [A_n(f_h;x) - f_h(x)] + [f_h(x) - f(x)] := K_1(x) + K_2(x) + K_3(x),$$

for $x\in R_0$, $n\in N$ and $h>0$. Applying Lemma 2 and (14), we get

$$\| K_1 \| \le \| f - f_h \| \le \omega_1(f;h).$$

By Theorem 1 and (15) it follows that

$$\| K_2 \| \le \frac{\sqrt{2}}{n} \| f_h \| \le \frac{\sqrt{2}}{n} h^{-1}\omega_1(f;h),$$

for $h>0$, $n\in N$. Hence from (16) and (15) we derive the inequality

$$\| A_n(f;r;\cdot) - f(\cdot) \| \le \left(2 + \frac{\sqrt{2}}{n} h^{-1}\right)\omega_1(f;h),$$

for every $n\in N$ and $h>0$. Choosing $h = \frac{\sqrt{2}}{n}$ for every fixed $n\in N$, we obtain

$$\| A_n(f;r;\cdot) - f(\cdot) | \le 3\omega_1\left(f;\frac{\sqrt{2}}{n}\right)$$

and we complete the proof of (12).

From Theorem 1 and Theorem 3 and by (6) we obtain

COROLLARY 1. *For every fixed $r\in R_2$ and $f\in C_0$ we have*

$$\lim_{n\to\infty} \| A_n(f;r;\cdot) - f(\cdot) \| = 0.$$

COROLLARY 2. *If $f\in C_0^1$ and $r\in R_2$, then*

$$\| A_n(f;r;\cdot) - f(\cdot) \| = O(1/n).$$

2.3. Finaly, we shall give the Voronovskaya type theorem for A_n.

THEOREM 3. *Let $f\in C_0^1$ and let $r\in R_2$ be fixed number. Then,*

(17) $$\lim_{n\to\infty} n\,\{A_n(f;r;x) - f(x)\} = f'(x)$$

for every $x\in R_0$.

PROOF. Let $x \in R_0$ be a fixed point. Then by the Taylor formula we have

$$f(t) = f(x) + f'(x)(t-x) + \varepsilon(t;x)(t-x)$$

for $t \in R_0$, where $\varepsilon(t) \equiv \varepsilon(t;x)$ is a function belonging to C_0 and $\varepsilon(x) = 0$. Hence by (7) and (8) we get

$$(18) \qquad A_n(f;r;x) = f(x) + f'(x)A_n(t-x;r;x) + A_n(\varepsilon(t)(t-x);r;x), \quad n \in N,$$

and by Hölder inequality

$$|A_n(\varepsilon(t)(t-x);r;x)| \le \left\{A_n(\varepsilon^2(t);r;x)\right\}^{1/2} \left\{A_n((t-x)^2;r;x)\right\}^{1/2}.$$

By Corollary 1 and by (2) we deduce that

$$\lim_{n\to\infty} A_n(\varepsilon^2(t);r;x) = \varepsilon^2(x) = 0.$$

From this and by Lemma 1 we get

$$(19) \qquad \lim_{n\to\infty} nA_n(\varepsilon(t)(t-x);r;x) = 0.$$

Using (19) and Lemma 1 to (18), we obtain the desired assertion (17).

REMARK 1. It is easily verified that analogous approximation properties in the space C_0 have the operators

$$\overline{A}_n(f;r;x) := e^{-(nx+1)^r} \sum_{k=0}^{\infty} \frac{(nx+1)^{rk}}{k!} n(nx+1)^{r-1} \int_{k/(n(nx+1)^{r-1})}^{(k+1)/(n(nx+1)^{r-1})} f(t)dt,$$

$f \in C_0$, $n \in N$, $x \in R_0$ and $r \in R_2$.

REMARK 2. In [1] was proved that if $f \in C_0$, then for the Szasz–Mirakyan operators S_n (defined by (1)) is satisfed the following inequality

$$|S_n(f;x) - f(x)| \le M_1 \omega_2\left(f;\sqrt{\frac{x}{n}}\right), \quad x \in R_0, \quad n \in N,$$

where $M_1 = const. > 0$ and $\omega_2(f;\cdot)$ is the modulus of smoothness defined by the formula

$$\omega_2(f;t) \equiv \omega_2(f;C_0;t) := \sup_{0\le h\le t} \|\Delta_h^2 f(\cdot)\|, \quad t \in R_0,$$

$\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$. In particular, if $f \in C_0^1$, then

$$|S_n(f;x) - f(x)| \le M_2 \sqrt{\frac{x}{n}},$$

for $x \in R_0$ and $n \in N$ $(M_2 = const. > 0)$.

Theorem 2 and Theorem 3 and Corollary 2 in our paper show that operators A_n, $n \in N$, give better degree of approximation of functions $f \in C_0$ and $f \in C_0^1$ than S_n.

REFERENCES

[1] M. Becker, Global approximation theorems for Szasz - Mirakjan and Baskakov operators in polynomial weight spaces, *Indiana Univ. Math. J.* 27(1)(1978), 127 - 142.

[2] R.A. De Vore, G.G. Lorentz, *Constructive Approximation*, Springer – Verlag, Berlin 1993.

(Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, Poland)

Received on 23.01.2002 and, in revised form, on 31.01.2002.