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## A CONTRIBUTION TO THE THEORY OF MODULAR SPACES OF SEQUENCES

ABSTRACT: For a given space  $T$  of all real sequences, non-negative matrix  $A = (a_{nm})$  and two sequences of convex  $\varphi$ -functions  $\varphi = (\varphi_m)$  and  $\psi = (\psi_m)$  we considered two modular spaces of sequences  $T_\varphi^*$  and  $T_\psi^*$ . This note contains theorems which determinate relationship between these spaces.

KEY WORDS: sequence spaces, modular spaces.

### 1. INTRODUCTION

In course of investigation of modular spaces there are considered concrete examples of modular spaces which may be applied in various problems of mathematical analysis. Modular spaces of sequences connected with strong summability have been introduced and investigated in papers of J. Musielak [4], J. Musielak and W. Orlicz [6], W. Orlicz [13], R. Taberski [14] and also myself.

Continuing the investigations of J. Musielak and W. Orlicz [7], in this paper we considered and investigated some modular spaces of sequences, which are connected with strong  $(A, \varphi)$ -summability and  $|A, \varphi|$ -summability, respectively (compare e.g. [7], [13], [15] and [17]). The main result of this paper are necessary conditions for inclusions of these spaces.

### 2. PRELIMINARIES

By a  $\varphi$ -function we understood a continuous non-decreasing function  $\varphi(u)$  defined for  $u \geq 0$  and such that  $\varphi(0) = 0$ ,  $\varphi(u) > 0$  for  $u > 0$  and  $\varphi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ . The symbol  $\varphi(|x|)$  means the function  $\varphi(|x(t)|)$ . Let  $\varphi$  be a convex  $\varphi$ -function satisfying conditions :

$$(0_1) \quad \frac{\varphi(u)}{u} \rightarrow 0 \quad \text{as } u \rightarrow 0_+, \quad (\infty_1) \quad \frac{\varphi(u)}{u} \rightarrow \infty \quad \text{as } u \rightarrow \infty,$$

Let  $\varphi = (\varphi_m)$  and  $\psi = (\psi_m)$  be any two non-decreasing sequences of convex  $\varphi$ -functions and let the functions  $\varphi_m$  and  $\psi_m$  ( $m = 1, 2, \dots$ ) satisfy conditions

( $0_1$ ) and ( $\infty_1$ ). The inverse  $\varphi$ -functions to the  $\varphi$ -functions  $\varphi_m$  and  $\psi_m$  will be denoted by  $\varphi_m^{-1}$  and  $\psi_m^{-1}$ , respectively.

In the following any relations between the sequences  $\varphi = (\varphi_m)$  and  $\psi = (\psi_m)$  we understand as these relations for  $\varphi$ -functions  $\varphi_m$  and  $\psi_m$  for all  $m$ .

Let  $A = (a_{nm})$  be a matrix which contains no column consisting of zeros only. In order to develop a theory of sequence modular spaces, we shall need in particular theorems some additional assumptions on the matrix  $A$ :

- a) is nonnegative i.e.  $a_{nm} \geq 0$  for  $n, m = 1, 2, \dots$ ,
- b) for an arbitrary positive integer  $n$  (or  $m$ ) there exists a positive integer  $n_0$  (or  $m_0$ ) such that  $a_{nm_0} \neq 0$  (or  $a_{n_0m} \neq 0$ ), respectively,
- c) there exist  $\lim_{n \rightarrow \infty} a_{nm} = 0$  for  $m = 1, 2, \dots$ ,
- d) there exist a positive constant  $K$  such that  $a_{n_1} + a_{n_2} + \dots \leq K$ , for all  $n$ ,
- d)  $A_1^\infty = \sup_n \sum_{m=1}^{\infty} a_{nm} < \infty$ ,
- e)  $A_m = \sup_n a_{nm} \rightarrow 0$  as  $m \rightarrow \infty$ ,
- f)  $\limsup_{m \rightarrow \infty} \frac{A_m}{A_{m+1}} = a < \infty$  as  $m \rightarrow \infty$ .

Let  $T, T_b, T_0, T_f$  denote spaces of all real sequences, bounded real sequences, real sequences convergent to zero and sequences with a finite number of elements different from zero, respectively. Sequences belonging to  $T$  will be denoted by  $x = (t_m), y = (s_m), x_k = (t_m^k), |x| = (|t_m|), 0 = (0)$  and  $x^n$  will mean the sequence  $t_1, t_2, \dots, t_n, 0, 0, \dots$ . Moreover, we shall write  $e_p, e^q, e_p^q$  for the following sequences:  $0, 0, \dots, 1, 0, \dots$  (with 1 at the  $p$ th place);  $1, 1, \dots, 1, 0, \dots$  (with 1 at the first  $q$  places);  $0, \dots, 0, 1, \dots, 1, 0, \dots$  (with 1 at the  $p$ th,  $(p+1)$ st, ...,  $(p+q-1)$ st places), respectively.

### 3. SPACES OF STRONGLY $(A, \varphi)$ AND $|A, \varphi|$ SUMMABLE SEQUENCES

In the space  $T_f$  we may define the following norm

$$\|x\|_\varphi^R = \inf \left\{ \varepsilon > 0 : \rho_\varphi^b \left( \frac{x}{\varepsilon} \right) \leq 1 \right\}$$

where

$$\rho_{\varphi}^b(x) = \sum_{m=1}^{\infty} \varphi_m(|t_m|)$$

is a modular in  $T_f$ .

It is well known that this norm is homogeneous and monotonic.

In the following, applying this norm we define the following class  $T_{\varphi}^b$  of sequences, strongly  $|A, \varphi|$ -summable to zero. For a given matrix  $A = (a_{nm})$  and  $\varphi$ -functions  $\varphi = (\varphi_m)$  we define a sequence  $\bar{x}^n$  by the formula

$$\bar{x}^n = \varphi_m^{-1}(a_{nm})t_m \quad \text{if } n \geq m \quad \text{and} \quad \bar{x}^n = 0 \quad \text{if } n < m.$$

Let  $T_{\varphi}^b$  denote the class of the sequences satisfying the condition

$$\lim_{n \rightarrow \infty} \|\bar{x}^n\| = 0.$$

Sequences  $x \in T$  which belong to  $T_{\varphi}^b$  are called strongly  $|A, \varphi|$ -summable to zero. Let us remark, that in the space  $T_{\varphi}^b$  we may introduce a norm by the formula

$$\|x\|_{\varphi}^b = \sup_n \|\bar{x}^n\|_{\varphi}^R$$

and moreover in the space  $T_f$  we can define another norm

$$\|x\|_{\varphi}^0 = \sup_y \sum_{m=1}^{\infty} |t_m| s_m,$$

where supremum is taken over all nonnegative sequence  $y = (s_m)$  satisfying the inequality

$$\sum_{m=1}^{\infty} \varphi_m^*(s_m) \leq 1,$$

Here  $\varphi = (\varphi_m)$  and  $\varphi^* = (\varphi_m^*)$  are two sequences of convex  $\varphi$ -functions and  $\varphi^*$  is a complementary to  $\varphi$  in the sense of Young.

For a given sequence of  $\varphi$ -functions  $\varphi = (\varphi_m)$  and the matrix  $A = (a_{nm})$  we adopt the following notation:

$$\sigma_n^{\varphi}(x) = \sum_{m=1}^{\infty} a_{nm} \varphi_m(|t_m|) \quad \text{for } n = 1, 2, \dots,$$

$$T_{\varphi}^0 = \{x \in T : \sigma_n^{\varphi}(x) < \infty \text{ for } n = 1, 2, \dots \text{ and } \lim_{n \rightarrow \infty} \sigma_n^{\varphi}(x) = 0\},$$

$$T_\varphi = \{x \in T : \lambda x \in T_\varphi^0 \text{ for an arbitrary } \lambda > 0\},$$

$$T_\varphi^* = \{x \in T : \lambda x \in T_\varphi^0 \text{ for a certain } \lambda > 0\}.$$

Sequences  $x$  belonging to  $T_\varphi^*$  are called strongly  $(A, \varphi)$ -summable to zero.

Obviously, the spaces and the norms defined above depend on the matrix  $A$ , however this dependence is not point out in the notation because in this paper we shall deal with a fixed matrix  $A$  only.

A list of the most interesting properties concerning the spaces  $T_\varphi^*$  and  $T_\varphi^b$  is presented for instance in [4-7], [15] and [17].

#### 4. PROPERTIES AND THEOREMS

**THEOREM 1.** *If  $T_\psi^b \subset T_\varphi^*$  then there exists  $\delta > 0$  satisfying the inequality*

$$(1.1) \quad \varphi(u)\psi(v) \leq \psi(\delta^{-1}uv)$$

for all  $u, v > 0$  such that  $uv < \delta$  and  $u \geq 1$ .

**PROOF.** We suppose that  $T_\psi^b \subset T_\varphi^*$  where  $\varphi$ -functions  $\varphi = (\varphi_m)$  and  $\psi = (\psi_m)$  are convex. Then there is an  $\eta > 0$  such that  $\|x\|_\psi^b < \eta$  implies  $\|x\|_\varphi^1 \leq 1$  for every  $x \in T_f$  (compare e.g. [5-7], [17], [19-21]).

Now take  $u, v > 0$  satisfying the condition

$$(1.2) \quad uv \leq \eta \psi_m^{-1}(1), \quad u \geq \eta$$

for all  $m$ . Then  $v \leq \psi_m^{-1}(1)$  for all  $m$  and there exist the integers  $p$  and  $q$  such that

$$(1.3) \quad \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) \leq v \leq \psi_m^{-1}(q A_p^q)$$

for  $m = p, p+1, \dots, p+q-1$ , where  $A_p^q = \sup_n \sum_{m=p}^{p+q-1} a_{nm}$ .

Hence applying (1.2) and (1.3) we have

$$(1.4) \quad u(\psi_m^{-1}(1))^{-1} \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) < \eta$$

for all  $m$ . In the following by the definition of  $\|\cdot\|_\psi^b$  and  $\|\cdot\|_\varphi^b$  we obtain

$$\|ue_p^q\|_\psi^b \leq u(\psi_p^{-1}(1))^{-1} \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) < \eta ,$$

$$\|ue_p^q\|_\varphi^1 = u[\varphi_p^{-1}((qA_p^q)^{-1})]^{-1} \leq 1$$

and

$$(1.5) \quad \varphi_p(u) \leq (qA_p^q)^{-1}.$$

Finally, be inequalities (1.3) and (1.5), we obtain

$$(1.6) \quad \varphi_p(u)\psi_p(v) \leq (qA_p^q)(qA_p^q)^{-1} = q.$$

Moreover, the inequality (1.4) give

$$\psi_p\left(\eta^{-1}u \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm})\right) \leq 1$$

and for a certain an index  $1 \leq r \leq q$  we have

$$\frac{1}{2} \leq r\psi_p\left[\frac{u}{\eta}\psi_p^{-1}(A_r)\right] < 1.$$

Consequently

$$(1.7) \quad q \leq 2qr\psi_p\left[\eta^{-1}u\psi_p^{-1}(A_r)\right] \leq 2qr\psi_p\left[\eta^{-1}u \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm})\right] \leq \psi_p(2q^2\eta^{-1}uv)$$

for all  $p$ , where  $u$  and  $v$  satisfy the condition (1.2).

By (1.6) and (1.7) we obtain (1.1), with  $\delta = \eta \min\{\psi_1^{-1}(1), (2q^2)^{-1}\} \leq \eta \min\{\psi_1^{-1}(1), 2^{-1}\}$ .

**THEOREM 2.** If  $T_\varphi^* \subset T_\psi^b$  then there exist a constant  $\delta > 0$  satisfying the inequality

$$(2.1) \quad \psi(\delta uv) \leq \varphi(u)\psi(v)$$

for all  $u, v > 0$  such that  $\varphi(u)\psi(v) \leq \delta$ ,  $\varphi(u) \geq 1$ .

**PROOF.** Let  $\varphi = (\varphi_m)$  and  $\psi = (\psi_m)$  be any two sequences of convex  $\varphi$ -functions. It is well known that if  $T_\varphi^* \subset T_\psi^b$  then there is an  $\eta > 0$  such that if  $\|x\|_\varphi^1 \leq \eta$  then  $\|x\|_\psi^b \leq 1$  for every  $x \in T_f$  (compare e.g. [4-7] and [17-21]).

Now, we choose  $u, v > 0$  such that

$$(2.2) \quad \varphi_m(u)\psi_m(v) \leq \eta, \quad \varphi_m(u) \geq 1$$

for every  $m$ . Thus  $\psi_m(v) \leq \eta$  for all  $m$  and there exist natural numbers  $p$  and  $q$  such that

$$(2.3) \quad \eta < (A_p)^{-1}\psi_m(v) \leq \eta q$$

and moreover

$$(2.4) \quad \frac{1}{2} \leq A_p^q \varphi_m(u) < 1$$

for all  $m$ . By (2.2) and (2.3) we obtain  $u < \varphi_m^{-1}((A_p)^{-1})$  for all  $m$ .

But  $\|u\eta e_p^q\|_\varphi^1 = u\eta\{\varphi_p^{-1}((A_p^q)^{-1})\}^{-1}$  and by (2.4) we have  $\|u\eta e_p^q\|_\psi^b \leq \eta$ .

Simple calculations lead to the following properties:

$$u\eta(q\psi_m^{-1}(1))^{-1}\psi_m^{-1}(A_p) \leq u\eta(q\chi\psi_m^{-1}(1))^{-1} \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) \leq \|u\eta e_p^q\|_\psi^b \leq 1,$$

$$u\eta < q\psi_m^{-1}(1) \left( \sup_n \sum_{n=p}^{p+q-1} \psi_m^{-1}(a_{nm}) \right)^{-1} \leq q\psi_m^{-1}(1)(\psi_m^{-1}(A_p))^{-1},$$

$$u\eta\psi_m^{-1}(A_p) \leq u\eta \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) \leq q\psi_m^{-1}(1),$$

$$\psi_m(q^{-1}u\eta\psi_m^{-1}(A_p)) \leq \psi_m \left( q^{-1}u\eta \sup_n \sum_{m=p}^{p+q-1} \psi_m^{-1}(a_{nm}) \right) \leq 1.$$

But by assumption

$$(2.5) \quad \psi_m^{-1}(A_p) \geq q\psi_m^{-1}(\eta q A_p)$$

for all  $m$ , where  $\eta q < 1$ . In the following by (2.3) and (2.5) we have

$$\psi_m(\eta uv) = \psi_m(\eta u\psi_m^{-1}(\eta q A_p)) \leq 1.$$

In consequence we obtain

$$(2.6) \quad \psi_m(\eta u v) \leq 1$$

for all  $m$ .

Applying the properties of the matrix  $A$  and the inequalities (2.3), (2.4) and (2.6) we get

$$\eta \leq (A_p)^{-1} \psi_m^{-1}(v) \leq q(A_p^q)^{-1} \psi_m(v) \leq 2q \psi_m(v) \varphi_m(u),$$

$$(2q)^{-1} \eta \leq \psi_m(v) \varphi_m(u),$$

$$(2q)^{-1} \eta \psi_m(\eta u v) \leq \psi_m(v) \varphi_m(u)$$

for all  $m$ . But  $\varphi$ -functions  $\psi_m$  are convex then we get

$$\psi_m((2q)^{-1} \eta^2 u v) \leq \psi_m(v) \varphi_m(u)$$

for all  $m$ .

Choosing  $2\delta = \eta^2 q^{-1}$ , we conclude the inequality (2.1).

Let us remark that above theorems are generalization of theorems due to Julian Musielak and Władysław Orlicz (see [7], pp. 135-139). In paper [7] there are considered and investigated some modular spaces connected with two  $\varphi$ -function  $\varphi$  and  $\psi$  and with the matrix  $A = (a_{nm})$ , where  $a_{nm} = \frac{1}{n}$  for  $n \geq m$  and  $a_{nm} = 0$  for  $n < m$ .

The paper [21] contains the theorems in which there are given the sufficient conditions for the inclusion of the spaces  $T_\psi^b$  and  $T_\varphi^*$ .

**THEOREM 3.**  $T_\psi^b \subset T_\varphi^*$  if and only if the condition (1.1) holds,

$T_\varphi^* \subset T_\psi^b$  if and only if the condition (2.1) holds.

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